

Three-charge black holes on a circle

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ABSTRACT: We study phases of five-dimensional three-charge black holes with a circle in their transverse space. In particular, when the black hole is localized on the circle we compute the corrections to the metric and corresponding thermodynamics in the limit of small mass. When taking the near-extremal limit, this gives the corrections to the finite entropy of the extremal three-charge black hole as a function of the energy above extremality. For the partial extremal limit with two charges sent to infinity and one finite we show that the first correction to the entropy is in agreement with the microscopic entropy by taking into account that the number of branes shift as a consequence of the interactions across the transverse circle. Beyond these analytical results, we also numerically obtain the entire phase of non- and near-extremal three- and two-charge black holes localized on a circle. More generally, we find in this paper a rich phase structure, including a new phase of three-charge black holes that are non-uniformly distributed on the circle. All these three-charge black hole phases are found via a map that relates them to the phases of five-dimensional neutral Kaluza-Klein black holes.

KEYWORDS: Black Holes in String Theory, D-branes, Gauge-gravity correspondence.

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1. Introduction

Three-charge black holes in five dimensions play a prominent role in string theory since the Bekenstein-Hawking entropy of these black holes can be explained by a microscopic counting of the degeneracies of states [1–5]. In particular, such three-charge black holes can be constructed by considering appropriate D-brane configurations in ten-dimensional string theory and reducing to five-dimensional supergravity. These three-charge black holes (as well as the related two-charge system) continue to receive a lot of attention in the literature, providing a fertile ground for further exploration of the microscopic origin of entropy in string theory (see e.g. the review [6, 7]) and understanding of the AdS₃/CFT₂ correspondence [8, 9].

One direction that we wish to explore in this paper is what happens to the entropy of such five-dimensional three-charge black holes when we compactify one of the directions, i.e. when we consider the asymptotic space to be four-dimensional Minkowski space times a circle. Obviously, for extremal black holes the BPS property ensures that we can localize them on the circle by considering an infinite array on the covering space of the circle. The entropy of this extremal configuration is the well-known constant $2\pi\sqrt{N_1 N_4 N_0}$.¹ However, when we move away from extremality the black holes will start to interact with each other and as a consequence the finite entropy will receive corrections.

One of the primary aims of this paper is the computation of these corrections in the limit of small mass (or, equivalently, large compact radius). In particular, in the near-extremal limit we find that the entropy including the first two corrections is given by

$$S = 2\pi\sqrt{N_1 N_4 N_0} \left(1 + \sqrt{\frac{\epsilon}{8}} + \frac{\epsilon}{16} + \mathcal{O}(\epsilon^{3/2}) \right) \quad (1.1)$$

where ϵ is a dimensionless quantity proportional to the energy above extremality. We also compute the corrected entropy when taking a partial extremal limit in which two of the charges are sent to infinity and one is kept finite. In this case, we are able to show that the first correction to the entropy is in agreement with the microscopic entropy formula [1, 5]

$$S = 2\pi\sqrt{N_1 N_4} \left[\sqrt{N_0} + \sqrt{\tilde{N}_0} \right] \quad (1.2)$$

when taking into account that as a consequence of the interactions across the transverse circle the number of D0 and anti-D0 branes will shift [10].

¹We generally work in this paper in a duality frame in which the extremal configuration is the F1-D0-D4 system, which is related to the P-D1-D5 system by a T-duality in the direction of the F1-string.

Beyond these analytical results, we will also numerically obtain the entire phase of non- and near-extremal three (and two-) charge black holes localized on a circle. More generally, we will find in this paper a rich phase structure, including a new phase of three-charge black holes that are non-uniformly distributed on the circle.

The method we employ in this paper builds on the one used in [11] (see also [12, 13]) where it was shown that any Kaluza-Klein black hole (see [14, 15] for reviews) in $d + 1$ dimensions ($4 \leq d \leq 9$) can be mapped to a corresponding brane solution of Type IIA/IIB String Theory and M-theory, following the procedure originally conceived in [16]. These are thermal excitations of extremal 1/2-BPS branes in String/M-theory with transverse space $\mathbb{R}^{d-1} \times S^1$, i.e. with a transverse circle. This gave a precise connection between the rich phase structure of Kaluza-Klein black holes and that of the corresponding brane. In particular, by considering the near-extremal limit of the latter, the thermodynamic behavior of the non-gravitational theories dual to near-extremal branes on a circle was obtained via the map.

As we will show in detail in this paper, by considering the particular case of Kaluza-Klein black holes in five dimensions ($d = 4$), the above map can be generalized to generate non-extremal three-charge brane configurations in Type IIA/IIB String Theory and M-theory from any five-dimensional Kaluza-Klein black hole. We will typically work in the duality frame where these charges are carried by the F1-D0-D4 system, and the solutions obtained are thermal excitations of the corresponding extremal 1/8-BPS brane system. When reduced on the spatial world-volume directions of these branes, we then obtain three-charge black holes in five-dimensional supergravity.

While the procedure used to generate these three-charge black holes is a standard generalization of the one-charge case, it turns out that many features and properties of the resulting map from five dimensional Kaluza-Klein black holes to three-charge black holes are very different in this case. At various points in the paper we will comment on these differences which seem highly connected to the fact that, contrary to the one-charge case, the extremal localized three-charge black hole has finite entropy.

Fortunately, much is known about the phase structure of five-dimensional Kaluza-Klein black holes. There are four known phases:

- (i) The uniform phase, i.e. the black string which is a four-dimensional Schwarzschild black hole times a circle. The horizon topology in this phase is $S^2 \times S^1$.
- (ii) The non-uniform phase, which is a static solution emerging from the uniform phase at the Gregory-Laflamme point [17, 18] where the black string is marginally unstable. The leading order behavior of this phase was found in [19–21] and very recently the entire² phase was numerically computed for five dimensions in ref. [24]. This phase also has horizon topology $S^2 \times S^1$ but the horizon is non-uniform along the direction of the circle.

²The non-uniform and localized phase are conjectured [22] to meet in a topology changing transition point. For Kaluza-Klein black holes in five and six dimensions this is highly supported by the data of [23, 24]. However, there is still a “gap” between the localized and non-uniform phase, so that the numerical branches may not be fully complete.

- (iii) The localized phase, which approaches the five-dimensional Schwarzschild black hole in the limit of zero mass. The first correction to this metric in the small-mass limit is known analytically [25] (see also [26, 27]) using the ansatz proposed in [28] and recently also the second correction [29, 30]³, while the entire phase was obtained numerically in [31, 32, 23]. The horizon topology in this phase is S^3 .
- (iv) The phase of bubble-black hole sequences [33–35], involving black objects held apart by Kaluza-Klein bubbles, where the topology of the black object is S^3 or $S^2 \times S^1$ depending on the position in the sequence. The bubbles support the contractible S^1 and provide the repelling force to keep the black holes in static equilibrium. For any such alternating sequence of bubbles and black holes the exact form of the solution is known.

The first three of the above three phases (not involving Kaluza-Klein bubbles) can be found in the $SO(3)$ symmetric ansatz proposed in [28] and proven in [36, 37], while the phase involving Kaluza-Klein bubbles occur in the generalized Weyl ansatz of ref. [33], with $SO(2)^3$ symmetry. Moreover, these phases can be summarized by plotting them in a two-dimensional phase diagram [38, 39, 37] where (relative) tension is plotted versus the mass.

By applying a sequence of boosts and U-dualities we show in this paper that for any five-dimensional Kaluza-Klein black hole, one obtains a corresponding five-dimensional three-charge black hole solution with a transverse circle. Each of the phases of Kaluza-Klein black holes described above thus directly maps onto a corresponding phase of non-extremal three-charge black holes and we can express the thermodynamic properties in terms of those of the seeding solution. Furthermore, the map becomes especially simple (as was the case for the one-charge branes considered in ref. [11]) when we take the near-extremal limit. This limit is also relevant for the dual CFT description of these brane systems and for our application to microstate counting. Various near-extremal limits are considered, in which three, two, and one of the charges are sent to infinity. We also discuss the special case when one of the three charges is zero. In this case the map of the thermodynamics from the neutral seeding solution to that of the near-extremal two-charge solution is also very simple, in analogy to the one-charge case considered in [11] (see [40] for a very short review). In the applications discussed in this paper we will primarily focus on the first three phases listed above. In particular, our emphasis will be on the localized phase, but we will also present the results for the uniform phase and new non-uniform phase.

The localized phase of the F1-D0-D4 system is particularly interesting, since by a T-duality in the direction of the F1-string this is dual to the P-D1-D5 system, and hence has a finite entropy in the extremal limit. For this case we can summarize our results as follows:

³Note that the second order correction to the thermodynamics of the localized phase was obtained in [30] for all d . However for the case of $d = 4$ in addition the corrections to the metric are known [29]. For simplicity we use in this paper only the first order corrected metric, but for the thermodynamics we include these second order corrections.

- Using the analytical results of [25] for the first order correction and the second order correction computed in [29], we find the corrected metric for non-extremal three-charge black holes on a circle and the corresponding thermodynamics in an expansion for small masses.
- We consider the near-extremal limit in which all three charges are sent to infinity, and give the corrected metric for near-extremal three-charge black holes on a circle and the corresponding corrected thermodynamics. In particular, we find the leading order correction (1.1) to the finite entropy of the F1-D0-D4 system due to the presence of the circle. We also use the numerical results of [23] to plot the thermodynamics of the entire localized phase of the F1-D0-D4 brane system on a circle.
- We furthermore consider special cases of the near-extremal limit in which one (or two) of the three charges are kept finite. In particular, for the case with one finite charge, we obtain the corrected entropy and show that the leading order correction is in agreement with a microscopic counting for the three-charge system on a circle in the dilute gas approximation. This is done by using the microscopic formula (1.2) and using the fact that, for fixed total energy, the charges are shifted due to the interaction across the transverse circle.
- The non-extremal three-charge system includes as a special case the two-charge D0-D4 system. By a T-duality this corresponds to the D1-D5 system with a transverse circle, so that the localized phase in this case is relevant for the dual two-dimensional CFT [8]. For this case we also obtain the corrected thermodynamics for small masses, and plot the thermodynamics of the entire localized phase. In the canonical ensemble, the corrections describe the small temperature expansion of the free energy around the conformal behavior $F \propto T^2$ of the two-dimensional CFT.

In parallel to these applications for the localized case, which from the point of view of microstate counting and dual CFT are most interesting, we also discuss the charged black hole solutions that are mapped from the uniform and non-uniform phase of five-dimensional Kaluza-Klein black holes. The latter phase generates a new phase of three-charge black holes non-uniformly distributed on the circle, which is interesting in its own right as a new static brane configuration in String/M-Theory, or equivalently in the corresponding five-dimensional supergravity.

When all three charges are non-zero, the uniform phase of F1-D0-D4 is T-dual to a P-D1-D5 system uniformly smeared on the transverse circle, which in turn can be dualized to a P-D2-D6 system. Correspondingly, the non-uniform phase is thus equivalent to the P-D1-D5 system non-uniformly distributed on the transverse circle. For this phase we can use the recent numerical data [24] for the non-uniform phase of five-dimensional Kaluza-Klein black holes to obtain plots of the thermodynamics. Likewise for the two-charge case, we have a new phase of D1-D5 branes non-uniformly distributed on the transverse circle. It turns out that in the near-extremal limit the uniformly smeared phase has Hagedorn behavior, with the non-uniform phase emerging at the Hagedorn point. More generally, it is

found that the thermodynamic behavior for the near-extremal two-charge case is completely analogous to that of the near-extremal NS5-brane recently considered in ref. [41].

The outline of the paper is as follows. In section 2 we show how to generate ten-dimensional three-charge configurations on a circle by applying a set of boosts and U-duality transformations to a general five-dimensional neutral Kaluza-Klein black hole. We will also refer to the latter as the seeding solution. We discuss how to measure the asymptotic quantities for these three-charge solutions and the corresponding mapping of physical quantities from the seeding solution to those of the non-extremal three-charge system.

In section 3 we first present some details on the seeding solutions, restricting to those that do not involve Kaluza-Klein bubbles, and which fall in the ansatz of [28]. As applications of the map of section 2 we then discuss the uniform, non-uniform and localized phase of three-charge black holes.

Section 4 defines the near-extremal limit that we take on the non-extremal three-charge solutions and discusses the corresponding physical quantities for the resulting near-extremal three-charge configurations when all three charges are sent to infinity. It turns out that for these configurations, tension along the transverse circle is proportional to the energy above extremality (unlike the one-charge case) and we present a general argument showing that this is the case for any system that involves a non-vanishing entropy in the extremal limit.

We then continue in section 5 by presenting the near-extremal phases that follow by applying the near-extremal limit of section 4 to the phases considered in section 3. This includes the uniform, non-uniform and localized phase of three-charge black holes on a circle.

Section 6 considers other near-extremal limits, in which we keep one or two of the three charges finite. In particular, the corrected thermodynamics for the localized phase is computed in these two cases. Furthermore, we discuss the special case where one of the three-charges is zero, i.e. non-extremal two-charge configurations with a transverse circle. In particular, the near-extremal limit of the D0-D4 system on a transverse circle is considered and the general map of the physical quantities. As an application we apply this map to the uniform, non-uniform and localized phase and present the resulting thermodynamics for each of these phases.

Finally, in section 7 we turn to the specific case of the partial extremal limit where two of the charges are sent to infinity and one is kept finite, focussing on the localized three-charge black hole. In this dilute gas regime we present a microstate counting argument, following ref. [10], that reproduces the first correction computed in section 6 using the map. We end with the conclusions and outlook in section 8.

A number of appendices is included. Appendix A gives some useful details on the boosts and U-dualities employed in section 2. Appendix B gives the relation between the asymptotics of the seeding solution and that of the three-charge solutions. Appendix C provides further details on our definition of electric masses and tensions.

2. Generating three-charge solutions from Kaluza-Klein black holes

In this section we will present the non-extremal three-charge solution generated from a

neutral Kaluza-Klein black hole in 4+1 dimensions. The neutral solution is referred to as the *seeding solution*. Further, we will see how to calculate the physical quantities of the new three-charge solution given the seeding solution.

2.1 Three-charge configuration on a circle

The system that we are interested in is a three-charge solution of Type IIA Supergravity that describes a thermal excitation of the 1/8-BPS configuration with an F1-string, D4-brane and a D0-brane. The configuration is a solution to the equation of motions of the action

$$I = \frac{1}{16\pi G_{10}} \int d^{10}x \sqrt{-g} \left(R - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{12} e^{-\phi} (dB)^2 - \frac{1}{2 \cdot 6!} e^{-\frac{1}{2}\phi} (dA_{(5)})^2 - \frac{1}{4} e^{\frac{3}{2}\phi} (dA_{(1)})^2 \right) \tag{2.1}$$

where ϕ is the dilaton field, B is the Kalb-Ramond two-form field and the $A_{(i)}$, $i = 5, 1$ are the gauge fields that couple to the D4-brane and the D0-brane, respectively. This is the low energy action of Type IIA String Theory when the only gauge fields present are the ones that correspond to the three extended objects that we are interested in. Note that the action is written in Einstein frame which we will use throughout.

The extremal 1/8-BPS solution of the action (2.1) is well known and can be found e.g. by using the harmonic rule [5, 42, 43]. The metric for such a solution consists of a world-volume part for the extended objects times a transverse space which will be four-dimensional if the extended objects do not intersect. It is important that the non-compact part of the transverse space has at least three dimensions in order to be able to measure asymptotic quantities. We are interested in solutions where the transverse space asymptotes to $\mathbb{R}^3 \times S^1$. The compact transverse circle gives rise to some interesting physics that we wish to explore.

The solution can be compactified on T^5 which is spanned by the spatial world-volume directions of the D4-brane and the F1-string. This gives a five-dimensional black hole with three charges that can be compared to the extremal solution of Strominger and Vafa [1]. We choose to consider the F1-D0-D4 configuration instead of the more traditional P-D1-D5 configuration because the background turns out to be simpler, it has diagonal metric and the objects do not share spatial world-volume directions. The systems are, however, T-dual.

2.2 Generating F1-D4-D0 solutions

The main idea of the present paper is that we can generate charged solutions of the type described above, starting from a neutral 4+1 dimensional Kaluza-Klein black hole. A $d+1$ dimensional static and neutral Kaluza-Klein black hole is defined here as a pure gravity solution that has at least one event horizon and asymptotes to d -dimensional Minkowski-space times a circle at infinity. The thermodynamics of these kinds of Kaluza-Klein black holes has been studied extensively recently and there are both numerical and analytical results available about their different phases (see the reviews [14, 15]). The key observation of the present paper is that we can translate information about the thermodynamics of the

well-studied seeding solutions into information about the thermal three-charge configuration in ten-dimensional supergravity with a transverse circle where little or nothing was known before.

Let us start with a static and neutral five-dimensional Kaluza-Klein black hole as a seeding solution. There is no dilaton and no gauge fields and we assume that the metric can be written in the form

$$ds_5^2 = -U dt^2 + \frac{L^2}{(2\pi)^2} V_{ab} dx^a dx^b \quad (2.2)$$

where U is a non-constant function that vanishes at the horizon(s) and asymptotes to one, and $V_{ab} dx^a dx^b$ describes a cylinder of circumference 2π in the asymptotic region. The metric should in other words asymptote to

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega_2^2 + dz^2 \quad (2.3)$$

where z is periodic with period L . We will refer to the dimensionful coordinates r and z when discussing the asymptotic behavior of the full metric.

By adding five flat dimensions x and u_i , $i = 1, \dots, 4$, to the neutral solution (2.2), and performing a series of boosts and U-dualities, we can construct a ten-dimensional solution of Type IIA Supergravity with three-charges. Each boost adds one charge which depends on the rapidity parameter α of the boost. The derivation is sketched in appendix A. The new solution has metric

$$ds_{10}^2 = H_1^{-\frac{3}{4}} H_4^{-\frac{3}{8}} H_0^{-\frac{7}{8}} \left(-U dt^2 + H_4 H_0 dx^2 + H_1 H_0 \sum_{i=1}^4 (du^i)^2 + H_1 H_4 H_0 \frac{L^2}{(2\pi)^2} V_{ab} dx^a dx^b \right), \quad (2.4)$$

a dilaton ϕ given by

$$e^{2\phi} = H_1^{-1} H_4^{-\frac{1}{2}} H_0^{\frac{3}{2}}, \quad (2.5)$$

a Kalb-Ramond field given by

$$B = \coth \alpha_1 (H_1^{-1} - 1) dt \wedge dx, \quad (2.6)$$

and gauge fields

$$A_{(5)} = \coth \alpha_4 (H_4^{-1} - 1) dt \wedge du^1 \wedge du^2 \wedge du^3 \wedge du^4, \quad (2.7)$$

$$A_{(1)} = \coth \alpha_0 (H_0^{-1} - 1) dt. \quad (2.8)$$

The H_a are harmonic functions of the transverse space, given by

$$H_a = 1 + (1 - U) \sinh^2 \alpha_a, \quad \text{for } a = 1, 4, 0. \quad (2.9)$$

This three-charge solution describes a non-extremal configuration with an F1-string, D4-brane and D0-brane. The label $a = 1, 4, 0$ refers to the type of object. In the following we will refer to the fields B , $A_{(5)}$ and $A_{(1)}$ collectively as A_a .

We can perform one more U-duality and map this solution into a configuration with three non-extremal M2-branes which is an excitation of a known 1/8-BPS state.

2.3 Measuring asymptotic quantities

The physical quantities of the configuration that can be measured asymptotically far away in the transverse space are the mass, the three different kind of charges, and, since we have a compact circle in the transverse space, the tension in the direction of the circle. The tension measures how hard the black hole pulls itself across the circle and has an interpretation as the binding energy of the black holes in the covering space of the circle. For extremal BPS black holes there is no net force between the black holes in the covering space because the electric force exactly cancels the gravitational force and the tension therefore is zero. For non-extremal black holes this is not the case and more interesting physics appears. We now show how information about all these quantities can be mapped from the neutral seeding solution to the new three-charge solution.

It is useful to assume that each spatial world-volume direction x and u^i is compactified on a circle of length L_x and L_{u^i} respectively. This gives us two rectangular tori with volumes V_1 and $V_4 = \prod_i L_{u^i}$. In the asymptotic region of the transverse space the components of the metric can be expanded to leading order⁴ in r

$$g_{tt} \simeq -1 + \frac{\bar{c}_t}{r}, \quad g_{zz} \simeq 1 + \frac{\bar{c}_z}{r} \tag{2.10}$$

$$g_{xx} \simeq 1 + \frac{\bar{c}_x}{r}, \quad g_{ii} \simeq 1 + \frac{\bar{c}_u}{r}, \quad \text{for } i = 1, \dots, 4, \tag{2.11}$$

and the dilaton and the non-vanishing components of the gauge fields are to leading order

$$\phi \simeq \frac{\bar{c}_\phi}{r}, \quad (A_a)_{t\dots} \simeq \frac{\bar{c}_{A_a}}{r}, \quad \text{for } a = 1, 4, 0. \tag{2.12}$$

The non-vanishing gauge field components are the same as in equations (2.6)–(2.8).

The total mass and the total tension along any compact direction can be found from the asymptotic behavior of the metric using the general formulae of [44] (see also [45, 46]). Following [11, 44] we find the mass and charge via

$$\bar{M} = \frac{\Omega_2}{gL} (2\bar{c}_t - \bar{c}_z - \bar{c}_x - 4\bar{c}_u) \tag{2.13}$$

$$Q_a = -\frac{\Omega_2}{gL} \bar{c}_{A_a} \tag{2.14}$$

where we have defined

$$g \equiv \frac{16\pi G_{10}}{V_1 V_4 L^2}. \tag{2.15}$$

The dimensionful parameter g is useful to define dimensionless mass and charge as

$$\bar{\mu} \equiv g\bar{M}, \quad q_a \equiv gQ_a. \tag{2.16}$$

⁴Note that the dependence on r really is in terms of $r^{(d-3)}$ where d is the number of transverse dimensions which in this case is 4.

The tensions in the compactified directions are found via [44]

$$L\bar{\mathcal{T}}_z = \frac{\Omega_2}{gL} (\bar{c}_t - 2\bar{c}_z - \bar{c}_x - 4\bar{c}_u) \quad (2.17)$$

$$L_x\bar{\mathcal{T}}_x = \frac{\Omega_2}{gL} (\bar{c}_t - \bar{c}_z - 2\bar{c}_x - 4\bar{c}_u) \quad (2.18)$$

$$L_{u^i}\bar{\mathcal{T}}_{u^i} = \frac{\Omega_2}{gL} (\bar{c}_t - \bar{c}_z - \bar{c}_x - 5\bar{c}_u). \quad (2.19)$$

The world-volume of the D0-brane has no spatial direction, but for calculational purposes we can still pretend that there is a “phantom” u^0 direction compactified on a circle of length L_0 . The tension in that direction would then be given by

$$L_0\bar{\mathcal{T}}_0 = \frac{\Omega_2}{gL} (\bar{c}_t - \bar{c}_z - \bar{c}_x - 4\bar{c}_u - \bar{c}_0). \quad (2.20)$$

This is useful when we discuss the contribution of the D0-brane to the electric mass in the next subsection.

2.4 Mapping of physical quantities

To find how the physical quantities of the charged solution are related to the original seeding solution we write the asymptotics of the metric of the seeding solution as

$$-g_{tt}^{\text{seed}} = U \simeq 1 - \frac{c_t}{r}, \quad g_{zz}^{\text{seed}} \simeq 1 + \frac{c_z}{r}. \quad (2.21)$$

Expressed in c_t and c_z the mass and tension of the seeding solution is given by [38, 39]

$$M = \frac{\Omega_2 L}{16\pi G_5} (2c_t - c_z), \quad \mathcal{T}_z = \frac{\Omega_2 L}{16\pi G_5} (c_t - 2c_z). \quad (2.22)$$

Correspondingly we have the dimensionless mass, μ , and tension, n

$$\mu = \frac{16\pi G_5}{L^2} M = \frac{\Omega_2}{L} (2c_t - c_z), \quad n = \frac{\mathcal{T}L}{M} = \frac{c_t - 2c_z}{2c_t - c_z}. \quad (2.23)$$

By plugging the asymptotics (2.21) into the solution (2.4)–(2.9) and expanding to first order, we find a relation between the expansion coefficients of the new solution and the seeding solution. This relation is spelled out in appendix B. Plugging into the equations for mass, charge and tension we get

$$\bar{M} = \frac{\Omega_2}{gL} (-c_z + c_t(2 + \sinh^2 \alpha_1 + \sinh^2 \alpha_4 + \sinh^2 \alpha_0)), \quad (2.24)$$

$$Q_a = \frac{\Omega_2}{gL} c_t \sinh \alpha_a \cosh \alpha_a \quad (2.25)$$

$$L\bar{\mathcal{T}}_z = \frac{\Omega_2}{gL} (c_t - 2c_z), \quad (2.26)$$

$$L_x\bar{\mathcal{T}}_x = \frac{\Omega_2}{gL} (-c_z + c_t(1 + \sinh^2 \alpha_1)), \quad (2.27)$$

$$L_{u^i}\bar{\mathcal{T}}_{u^i} = \frac{\Omega_2}{gL} (-c_z + c_t(1 + \sinh^2 \alpha_4)), \quad (2.28)$$

$$L_0\bar{\mathcal{T}}_0 = \frac{\Omega_2}{gL} (-c_z + c_t(1 + \sinh^2 \alpha_0)). \quad (2.29)$$

Thus we have found how the physical quantities of the charged solution are given in terms of the boost parameters α_a and the two independent quantities c_t and c_z of the neutral solution.

The electric mass and tensions

The electric part of the mass and tensions are simply defined as the parts that go to zero when the charges Q_a vanish. We can directly read from equations (2.24) and (2.25) that

$$M^{\text{el}} = \sum_a \frac{\Omega_2}{gL} c_t \sinh^2 \alpha_a. \tag{2.30}$$

We see that the electric mass consists of a three parts — one for each of the charged objects. Thus it is natural to define the electric mass M_a^{el} corresponding to the charge Q_a as

$$M_a^{\text{el}} = L_a (\bar{\mathcal{T}}_a)^{\text{el}} = \frac{\Omega_2}{gL} c_t \sinh^2 \alpha_a, \tag{2.31}$$

for $a = 1, 4, 0$ ($x, u^i, 0$). In this notation a can either label the type of object or one of the spatial world-volume directions of the corresponding object.

Note that the electric mass can be written as

$$M_a^{\text{el}} = \nu_a Q_a \tag{2.32}$$

where

$$\nu_a = \tanh \alpha_a \tag{2.33}$$

is the chemical potential. The chemical potential can also be measured as $\nu_a = -A_a|_{\text{Horizon}}$ which by setting $U = 0$ in (2.6)–(2.8) gives the same result.

We also note that the electric part of the tension \mathcal{T}_z is zero. To see how all of this follows from the harmonic function rule in generality see appendix C where the electric masses and tensions are calculated in detail.

The mapping of dimensionless quantities

In the Kaluza-Klein black hole literature it is customary to define a relative tension n as the total tension divided by the total mass. Branches of different types of static solutions are then plotted on a (μ, n) phase diagram, where μ is the dimensionless mass. This kind of phase diagrams will be discussed further in section 3.

For the charged black holes under consideration here, we define the relative tension along the z direction as [11]

$$\bar{n} \equiv \frac{L \bar{\mathcal{T}}_z}{\bar{M} - M^{\text{el}}} = \frac{c_t - 2c_z}{2c_t - c_z} \tag{2.34}$$

and the relative tension along each of the world-volume directions as

$$\bar{n}_a \equiv \frac{L_a (\bar{\mathcal{T}}_a - \bar{\mathcal{T}}_a^{\text{el}})}{\bar{M} - M^{\text{el}}} = \frac{c_t - c_z}{2c_t - c_z}. \tag{2.35}$$

Note that we have chosen to subtract the electrical contribution in these definitions. In equations (2.34)–(2.35) we have also written the relative tensions in terms of the seeding parameters c_t and c_z .

Writing the physical parameters $\bar{\mu}$, \bar{n} and q_a in terms of the original quantities μ , n of the seeding solution and the boost parameters α_a gives

$$\bar{\mu} = \mu \left(1 + \frac{2-n}{3} (\sinh^2 \alpha_1 + \sinh^2 \alpha_4 + \sinh^2 \alpha_0) \right), \quad (2.36)$$

$$q_a = \mu \frac{2-n}{3} \sinh \alpha_a \cosh \alpha_a, \quad \text{for } a = 1, 4, 0 \quad (2.37)$$

and finally

$$\bar{n} = n, \quad \bar{n}_a = \frac{1+n}{3}. \quad (2.38)$$

We can solve equation (2.37) for $\cosh \alpha_a$ and get

$$\cosh \alpha_a = \sqrt{\frac{1}{2} \left(1 + \frac{1}{b_a} \sqrt{1 + b_a^2} \right)} \quad (2.39)$$

with

$$b_a \equiv \frac{2-n}{6} \frac{\mu}{q_a}. \quad (2.40)$$

Given the values of the three charges, the map between the mass and relative tension of the neutral and charged solutions can therefore be written as

$$\bar{n} = n, \quad (2.41)$$

$$\bar{\mu} = \sum_a q_a + \frac{1}{2} \mu n + \frac{(2-n)\mu}{6} \sum_a \frac{b_a}{1 + \sqrt{1 + b_a^2}}. \quad (2.42)$$

This map from neutral to three-charge Kaluza-Klein black holes is one of our main results.

There are a few things to notice here. The neutral seeding solutions always have $\mu \geq 0$ and $0 \leq n \leq 2$ and therefore we see that for fixed q_a the mass $\bar{\mu}$ is bounded from below by $\sum_a q_a$. This is to be expected for a charged black hole. An object with mass smaller than its charge results in a naked singularity rather than a black hole. We therefore define the *energy above extremality* to be the mass minus the sum of the charges. Later we will see that $b_a \rightarrow 0$ in the near-extremal limit, and therefore

$$\epsilon \equiv \bar{\mu} - \sum_a q_a \rightarrow \frac{1}{2} \mu n \quad (2.43)$$

in that limit. The mass of the charged black hole can also be written as $\bar{\mu} = \mu + \sum_a \nu_a q_a$, where $\nu_a = \tanh \alpha_a$ is the chemical potential from equation (2.32).

If the seeding solution has a single connected horizon we can find its temperature T and entropy S from the metric. In this paper we will mostly work with rescaled temperature and entropy which are defined for the seeding solution as [14]

$$\mathfrak{t} = LT, \quad \mathfrak{s} = \frac{16\pi G_5}{L^3} S. \quad (2.44)$$

For the three-charge solution we define the rescaled temperature and entropy analogously by

$$\bar{t} = L\bar{T}, \quad \bar{s} = \frac{g}{L}\bar{S}, \quad (2.45)$$

where g is given in equation (2.15). These quantities are calculated at the horizon where $U = 0$ and thus $H_a = \cosh^2 \alpha_a$. It is easy to see from the metric (2.4) that the three-charge solution therefore has temperature and entropy given by

$$\bar{t} = t / \cosh \alpha_1 \cosh \alpha_4 \cosh \alpha_0, \quad (2.46)$$

$$\bar{s} = s \cosh \alpha_1 \cosh \alpha_4 \cosh \alpha_0. \quad (2.47)$$

The factors of $\cosh \alpha_a$ cancel when we multiply the temperature and entropy and therefore the product

$$\bar{t}\bar{s} = ts \quad (2.48)$$

remains fixed. The generalized Smarr formula from [44] gives a relation between the temperature, entropy, and the gravitational mass for our three-charge solution in terms of the relative tension

$$\bar{t}\bar{s} = \frac{2 - \bar{n}}{3} (\bar{\mu} - \mu^{\text{el}}). \quad (2.49)$$

3. Non-extremal three-charge black holes on a circle

In this section we apply the map, that was found in section 2, to obtain three-charge black holes on a circle from neutral Kaluza-Klein black holes. We restrict ourselves to neutral black holes without Kaluza-Klein bubbles, and we review the ansatz for the metric for this class of Kaluza-Klein black holes. We then go on to describe the three different phases that we obtain for three-charge black holes on a circle, using the map.

3.1 The neutral seeding solutions

We have seen in section 2 that we can transform five-dimensional static and neutral black hole solutions to three-charge solutions via boosts and U-dualities. We now specify which class of solutions that we consider as the seeding solution, i.e. what solutions we will map to three-charge solutions.

We consider the Kaluza-Klein black holes in five-dimensions (see [14, 15] for reviews). These are the solutions asymptoting to four-dimensional Minkowski space times a circle $\mathcal{M}^4 \times S^1$, i.e. to Kaluza-Klein space. As reviewed in section 2 we can measure the rescaled mass μ and relative tension n for this class of solutions. In general we have solutions with $0 \leq n \leq 2$. However, the solutions in the range $1/2 \leq n \leq 2$ have Kaluza-Klein bubbles [35, 14] and we will not consider these here. Restricting ourselves to solutions with $0 \leq n \leq 1/2$ we have three types of solutions.

- The uniform black string. The metric of a uniform black string is simply a Schwarzschild black hole times a circle. The horizon topology is $S^2 \times S^1$.

- The non-uniform black string. This solution is a black string which is non-uniform around a circle. The horizon topology is $S^2 \times S^1$. The non-uniform black string solutions have been found numerically in [24] (see also [19–21]).
- Localized black holes. These are black holes which are localized on the circle, i.e. the horizon is so small it is not connected around the circle. The horizon topology is S^3 . Analytical results have been found in [25, 26, 29, 30]. Numerical results have been found in [23] (see also [31, 32]).

The metric for all three types of solutions can be written in the ansatz [28],

$$ds^2 = -f dt^2 + \frac{L^2}{(2\pi^2)} \left[\frac{A}{f} dR^2 + \frac{A}{K^2} dv^2 + KR^2 d\Omega_2^2 \right], \quad f = 1 - \frac{R_0}{R}. \quad (3.1)$$

In this ansatz the metric is specified by the two functions $A(R, v)$ and $K(R, v)$. One has furthermore that $A(R, v)$ can be found explicitly in terms of $K(R, v)$ [28]. The horizon is located at $R = R_0$. See [28, 36, 37, 47] for more on this ansatz.

We have displayed the (μ, n) phase diagram for $0 \leq n \leq 1/2$ with the three types of solutions⁵ in figure 1. We will review these three phases further as needed for describing the three-charge phases.

3.2 The ansatz for three-charge black holes on a circle

Using the map described in section 2.2, we map the ansatz (3.1) for neutral Kaluza-Klein black holes to the following ansatz for three-charge black holes:

$$ds_{10}^2 = H_1^{-\frac{3}{4}} H_4^{-\frac{3}{8}} H_0^{-\frac{7}{8}} \left[-f dt^2 + H_4 H_0 dx^2 + H_1 H_0 \sum_{i=1}^4 (du^i)^2 + H_1 H_4 H_0 \frac{L^2}{(2\pi)^2} \left(\frac{A}{f} dR^2 + \frac{A}{K^2} dv^2 + KR^2 d\Omega_2^2 \right) \right], \quad (3.2)$$

with

$$f = 1 - \frac{R_0}{R}, \quad H_a = 1 + \frac{R_0 \sinh^2 \alpha_a}{R}, \quad \text{for } a = 1, 4, 0, \quad (3.3)$$

and with the dilaton and the gauge fields still given by (2.5)–(2.8).

Using the ansatz for three-charge black holes on a circle (3.2), (3.3), (2.5)–(2.8), we can work out the following explicit physical parameters (assuming a single connected horizon)

$$\begin{aligned} \bar{\mu} &= 2R_0 \left(2 - \chi + \sum_a \sinh^2 \alpha_a \right), \quad \bar{n} = \frac{1-2\chi}{2-\chi}, \\ \bar{\mathfrak{t}} &= \frac{1}{2\sqrt{A_h} R_0 \prod_a \cosh \alpha_a}, \quad \bar{\mathfrak{s}} = 4\sqrt{A_h} R_0^2 \prod_a \cosh \alpha_a, \\ q_a &= 2R_0 \sinh \alpha_a \cosh \alpha_a, \quad \nu_a = \tanh \alpha_a, \quad \bar{n}_a = \frac{1-\chi}{2-\chi}, \quad a = 1, 4, 0, \end{aligned} \quad (3.4)$$

⁵Note that for the non-uniform and localized phase there are extra phases in the form of copies since the solutions can be copied k times on the circle [48, 37, 11]. At the level of solutions this means that given the solution (3.1) we obtain for any $k = 2, 3, \dots$ a new solution with $A'(R, v) = A(kR, kv)$, $K'(R, v) = K(kR, kv)$ and $R'_0 = R_0/k$. As a consequence we have also copies for the corresponding localized and non-uniform phase of three-charge black holes obtained via our map.

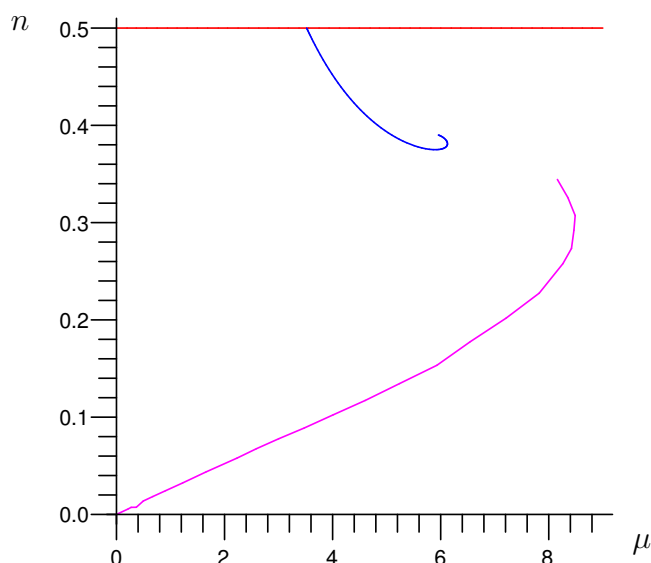


Figure 1: Diagram with μ versus n for the uniform black string (red), non-uniform black string (blue) and localized black hole phase (magenta) for five-dimensional Kaluza-Klein black holes, using numerical results of [23, 24].

where χ is defined from the asymptotic behavior of $K(R, v)$ as

$$K(R, v) = 1 - \chi \frac{R_0}{R} + \mathcal{O}(R^{-2}) \quad (3.5)$$

for $R \gg 1$, and where

$$A_h \equiv A(R, v)|_{R=R_0} . \quad (3.6)$$

Note that one can show that $\partial_v A(R, v) = 0$ on the horizon $R = R_0$ [28]. It is straightforward to see from the thermodynamics (3.4) along with the relation $\mu^{\text{el}} = \sum_a \nu_a q_a$ that we get the Smarr formula (2.49).

3.3 The uniform and non-uniform phases

The uniform phase corresponds to the F1-D0-D4 system smeared uniformly on a transverse circle. The supergravity solution for this is easily obtained by putting $A = K = 1$ in the ansatz (3.2), (3.3), (2.5)–(2.8). The thermodynamics is obtained from (3.4) setting $\chi = 0$ and $A_h = 1$, giving

$$\bar{\mu} = 2R_0 \left(2 + \sum_a \sinh^2 \alpha_a \right), \quad \bar{\tau} = \frac{1}{2R_0 \prod_a \cosh \alpha_a}, \quad \bar{s} = 4R_0^2 \prod_a \cosh \alpha_a \quad (3.7)$$

with q_a and ν_a as given in (3.4). We have furthermore that the relative tension is $\bar{n} = 1/2$. The uniform phase is mapped from the neutral uniform black string in five dimensions. The horizon topology for our non-extremal F1-D0-D4 system in the uniform phase is $T^5 \times S^2 \times S^1$, where the T^5 is along the charged directions.

The non-uniform phase of our F1-D0-D4 system is a phase in which the F1-D0-D4 is still distributed on the transverse circle without gaps, but with the distribution being non-uniform along the circle direction. The horizon topology is therefore the same as for the uniform phase: $T^5 \times S^2 \times S^1$. The non-uniform phase is mapped by (2.4)–(2.9) from the neutral non-uniform black string phase. From this fact it is easy to see using eq. (2.42) that the non-uniform phase emanates from the uniform phase in a critical point corresponding to the mass

$$\bar{\mu}_c = \sum_a q_a + x + x^2 \sum_a (q_a + \sqrt{x^2 + q_a^2})^{-1}, \quad x \simeq 0.88 \quad (3.8)$$

which is mapped from the Gregory-Laflamme mass $\mu_{\text{GL}} = 3.52$ of the five-dimensional neutral uniform black string [19, 21, 14] using that $x = \mu_{\text{GL}}/4$. We expect that the uniform phase is unstable to linear perturbations for masses $\bar{\mu} < \bar{\mu}_c$. Indeed, one should be able to find the explicit unstable mode using the methods of [13, 49] where the unstable mode of one-charge smeared branes were constructed by transforming the unstable mode for the neutral black string.

As reviewed in section 3.1, the neutral non-uniform black string solution is obtained numerically in [24]. This numerical solution can then be mapped to a numerical solution for the non-uniform phase of the F1-D0-D4 system, using either the map (2.4)–(2.9), or the ansatz (3.2), (3.3), (2.5)–(2.8). Similarly we can map the physical quantities using the results of section 2.4. We do not go into details with this, since the qualitative features of the mapped solution are highly similar to that of the neutral seeding solution. Only in the near-extremal limits that we consider in sections 4 and 5, one sees significant differences in the qualitative behavior. However, it is interesting to find the slope of the non-uniform phase in the $(\bar{\mu}, \bar{n})$ diagram near the critical point $(\bar{\mu}_c, 1/2)$ since this in a simple way can tell us about some of the features of the non-uniform phase as we change the charges. Using that the neutral non-uniform black string has the slope $n \simeq 1/2 - \gamma(\mu - \mu_{\text{GL}})$ ($\gamma \simeq 0.14$) near the Gregory-Laflamme point $(\mu, n) = (\mu_{\text{GL}}, 1/2)$ [19, 21, 14], we get the slope

$$\bar{n} \simeq \frac{1}{2} - \eta(\bar{\mu} - \bar{\mu}_c), \quad \eta = \gamma \left[\frac{1}{4} - 2\gamma x + x \left(\frac{1}{4} + \frac{2}{3}\gamma x \right) \sum_a \frac{2q_a \sqrt{x^2 + q_a^2} + x^2 + 2q_a^2}{\sqrt{x^2 + q_a^2}(q_a + \sqrt{x^2 + q_a^2})} \right]^{-1} \quad (3.9)$$

for $0 \leq \bar{\mu} - \bar{\mu}_c \ll 1$, when considering fixed charges q_a .

3.4 The localized phase

The localized phase of the F1-D0-D4 system corresponds to having the horizon of F1-D0-D4 localized on the transverse circle, such that the horizon is not connected across the circle. The horizon topology is therefore $S^3 \times T^5$, where T^5 is along the charged directions.

If we consider the case in which the size of the horizon is very small compared to the circumference of the transverse circle, we can write down an analytic expression for the metric using [25] (see [26, 29, 30] for more analytical results for such black holes). To this end, we should slightly modify the ansatz (3.2), (3.3), (2.5)–(2.8) by expressing it instead in the new coordinates $\tilde{\rho}$ and $\tilde{\theta}$ defined by

$$2R = \tilde{\rho}^2, \quad v = \pi - 2\tilde{\theta} + 2 \sin \tilde{\theta} \cos \tilde{\theta}. \quad (3.10)$$

We thus have the relation $\rho_0^2 = 2R_0$ for the horizon radius. With this, we can rewrite the ansatz (3.2), (3.3), (2.5)–(2.8) as

$$ds_{10}^2 = H_1^{-\frac{3}{4}} H_4^{-\frac{3}{8}} H_0^{-\frac{7}{8}} \left[-f dt^2 + H_4 H_0 dx^2 + H_1 H_0 \sum_{i=1}^4 (du^i)^2 + H_1 H_4 H_0 \frac{L^2}{(2\pi)^2} \left(\frac{\tilde{A}}{f} d\tilde{\rho}^2 + \frac{\tilde{A}}{\tilde{K}^2} \tilde{\rho}^2 d\tilde{\theta}^2 + \tilde{K} \tilde{\rho}^2 \sin^2 \tilde{\theta} d\Omega_2^2 \right) \right], \quad (3.11)$$

with

$$f = 1 - \frac{\rho_0^2}{\tilde{\rho}^2}, \quad H_a = 1 + \frac{\rho_0^2 \sinh^2 \alpha_a}{\tilde{\rho}^2}, \quad \text{for } a = 1, 4, 0. \quad (3.12)$$

Using the results of [25] one can now write down the full solution for the case in which the horizon is very small, i.e. for $\rho_0 \ll 1$. For simplicity, we discuss here only the part concerning the solution near the horizon, but it is straightforward to use the map to find the full solution. From [25] we get for $\rho_0 \leq \tilde{\rho} \ll 1$ that

$$\tilde{A}^{-\frac{1}{3}} = \tilde{K}^{-1} = \frac{1 - w^2}{w} \frac{\tilde{\rho}^2}{\rho_0^2} + w, \quad w = 1 + \frac{1}{24} \rho_0^2 + \mathcal{O}(\rho_0^4). \quad (3.13)$$

From [25] we have furthermore that $\chi = \frac{1}{2} - \frac{1}{32} \rho_0^2 + \mathcal{O}(\rho_0^4)$ and $\tilde{A}_h = 1 + \frac{1}{8} \rho_0^2 + \mathcal{O}(\rho_0^4)$, with $\tilde{A}_h = \tilde{A}|_{\tilde{\rho}=\rho_0}$. Using then that $A = \tilde{A}/\tilde{\rho}^2$ together with the thermodynamics (3.4), we can get the thermodynamics for $\rho_0 \ll 1$. However, before writing down this thermodynamics, we note that the second order correction has been obtained in [29, 30] which we can translate to our notation as⁶

$$\chi = \frac{1}{2} - \frac{1}{32} \rho_0^2 + \mathcal{O}(\rho_0^6), \quad \sqrt{\tilde{A}_h} = 1 + \frac{1}{16} \rho_0^2 + \frac{1}{512} \rho_0^4 + \mathcal{O}(\rho_0^6). \quad (3.14)$$

This now gives the thermodynamics

$$\begin{aligned} \bar{\mu} &= \rho_0^2 \left(\frac{3}{2} + \frac{1}{32} \rho_0^2 + \mathcal{O}(\rho_0^6) \right) + \sum_a \sinh^2 \alpha_a, \quad \bar{n} = \frac{1}{24} \rho_0^2 - \frac{1}{1152} \rho_0^4 + \mathcal{O}(\rho_0^6), \\ \bar{\mathfrak{t}} &= \frac{1 - \frac{1}{16} \rho_0^2 + \frac{1}{512} \rho_0^4 + \mathcal{O}(\rho_0^6)}{\rho_0 \prod_a \cosh \alpha_a}, \quad \bar{\mathfrak{s}} = \rho_0^3 \left(1 + \frac{1}{16} \rho_0^2 + \frac{1}{512} \rho_0^4 + \mathcal{O}(\rho_0^6) \right) \prod_a \cosh \alpha_a, \\ q_a &= \rho_0^2 \cosh \alpha_a \sinh \alpha_a, \quad \nu_a = \tanh \alpha_a, \quad \bar{n}_a = \frac{1}{3} + \frac{1}{72} \rho_0^2 - \frac{1}{3456} \rho_0^4 + \mathcal{O}(\rho_0^6). \end{aligned} \quad (3.15)$$

This is thus the thermodynamics of a small three-charge black hole localized on a circle.

4. Near-extremal three-charge black holes on a circle

We now consider the near-extremal limit of our three-charge black holes on a circle. This is an interesting limit in view of the microscopic counting of entropy and, more generally, in the context of the dual CFT. In this section we will see how to define this limit and its consequences for the physical quantities.

⁶Note that this follows from the slope of $n(\mu) = \mu/6^2 - \mu^2/6^4 + \mathcal{O}(\mu^3)$ for the neutral seeding solution, as one can see from the fact that $\mu = (2 - \chi)\rho_0^2$ and $n = (1 - 2\chi)/(2 - \chi)$.

4.1 The near-extremal limit

There are two ways to take the near-extremal limit. One can either keep the charges fixed and send the temperature to zero or keep the temperature fixed and send the charges to infinity. Since we are interested in the thermodynamics of the near-extremal black hole, it is natural for us to take the second option.

In order to retain the non-trivial physics related to the presence of the circle we want to take the limit in such a way that the size of the circle has the same scale as the energy above extremality. This means that the metric components multiplying dt^2 and $V_{ab}dx^a dx^b$ should scale in the same way. From the metric in eq. (2.4) we therefore require

$$\lim_{L \rightarrow 0} H_1 H_4 H_0 \left(\frac{L}{2\pi} \right)^2 = \text{finite}. \quad (4.1)$$

A natural way to achieve (4.1) is to demand

$$\lim_{L \rightarrow 0} H_a \left(\frac{L}{2\pi} \right)^{2\gamma_a} = \text{finite}, \quad a = 1, 4, 0, \quad (4.2)$$

where $\gamma_a \geq 0$ and $\gamma_1 + \gamma_4 + \gamma_0 = 1$. In this section we consider only the case when all γ_a are non-vanishing, postponing other limits to the next section. For $\gamma_a > 0$ the requirement of equation (4.2) means that

$$H_a = 1 + (1 - U) \sinh^2 \alpha_a \rightarrow \infty \quad (4.3)$$

so that the boost parameters α_a must go to infinity (for simplicity we will take α_a to be positive). In order to see what this means for the charges, it is convenient to introduce rescaled coordinates on the transverse space⁷

$$\hat{r} \equiv \frac{2\pi}{L} r, \quad \hat{z} \equiv \frac{2\pi}{L} z, \quad (4.4)$$

and corresponding rescaled expansion coefficients for the seeding metric (2.21)

$$-g_{tt}^{\text{seed}} \simeq 1 - \frac{\hat{c}_t}{\hat{r}}, \quad g_{zz}^{\text{seed}} \simeq 1 + \frac{\hat{c}_z}{\hat{r}}. \quad (4.5)$$

These coefficients are more appropriate in the near-extremal limit since they remain finite. We can now write the dimensionless charges (2.16) as

$$q_a = \frac{\Omega_2}{2\pi} \hat{c}_t \sinh \alpha_a \cosh \alpha_a \quad (4.6)$$

and from this expression it is apparent how the charges diverge. Notice that in order to satisfy the condition (4.1), we should keep fixed the parameters

$$\ell_a \equiv L^{\gamma_a} \sqrt{q_a} = \left(\frac{\Omega_2 \hat{c}_t}{2\pi} \right)^{1/2} L^{\gamma_a} \sqrt{\sinh \alpha_a \cosh \alpha_a}. \quad (4.7)$$

⁷Note that the coordinates of the ansatz (3.1) approach these dimensionless coordinates in the asymptotic region, $R \rightarrow \hat{r}$ and $v \rightarrow \hat{z}$.

To get a finite solution for the metric, the gauge fields and the dilaton in the near-extremal limit, we must rescale the fields with appropriate powers of $L/2\pi$ and the powers will depend on γ_a . This rescaling should be a symmetry of the action (2.1). It is fairly easy to check that the following scaling works

$$e^{2\phi^{\text{new}}} = \left(\frac{L}{2\pi}\right)^{-2\gamma_1 - \gamma_4 + 3\gamma_0} e^{2\phi^{\text{old}}}, \quad (4.8)$$

$$g_{tt}^{\text{new}} = \left(\frac{L}{2\pi}\right)^{-\frac{3}{2}\gamma_1 - \frac{3}{4}\gamma_4 - \frac{7}{4}\gamma_0} g_{tt}^{\text{old}}, \quad g_{xx}^{\text{new}} = \left(\frac{L}{2\pi}\right)^{-\frac{3}{2}\gamma_1 - \frac{3}{2}\gamma_4 + \frac{1}{4}\gamma_0} g_{xx}^{\text{old}}, \quad (4.9)$$

$$g_{u_i u_i}^{\text{new}} = \left(\frac{L}{2\pi}\right)^{\frac{1}{2}\gamma_1 - \frac{3}{4}\gamma_4 + \frac{1}{4}\gamma_0} g_{u_i u_i}^{\text{old}}, \quad g_{rr}^{\text{new}} = \left(\frac{L}{2\pi}\right)^{\frac{1}{2}\gamma_1 + \frac{5}{4}\gamma_4 + \frac{1}{4}\gamma_0 + 2} g_{rr}^{\text{old}}, \quad (4.10)$$

$$A_a^{\text{new}} = \left(\frac{L}{2\pi}\right)^{-2\gamma_a} A_a^{\text{old}}, \quad G_{10}^{\text{new}} = \left(\frac{L}{2\pi}\right)^2 G_{10}^{\text{old}}. \quad (4.11)$$

The components of the metric for the other transverse directions are rescaled in the same way as g_{rr} . The choice of powers of $L/2\pi$ is unique and can be found by first requiring the gauge fields, B field, and dilaton to be finite in the limit. Next the scalings of the metric is found by requiring all the terms in the action to scale in the same way. Finally, the scaling of Newton's constant is found by requiring the scaling to be a symmetry of the action.

The choice of gauge in equations (2.6)–(2.8) is not convenient in the near-extremal limit because the constant term will be dominant. But we are free to change the gauge by adding the constant $\coth \alpha_a$ to our old A_a^{old} before taking the limit. This gives

$$A_a^{\text{old}} = \coth \alpha_a H_a^{-1} = \left(\frac{L}{2\pi}\right)^{2\gamma_a} \coth \alpha_a \hat{H}_a^{-1} \quad (4.12)$$

where we have defined

$$\hat{H}_a \equiv \lim_{L \rightarrow 0} \left(\frac{L}{2\pi}\right)^{2\gamma_a} H_a. \quad (4.13)$$

By construction, \hat{H}_a is finite in the near-extremal limit and from equation (4.11) we see that after rescaling A_a^{new} will be finite as well.

To summarize, the particular near-extremal limit that we are interested in can be defined as

$$L \rightarrow 0, \quad \alpha_a \rightarrow \infty, \quad \ell_a \equiv L^{\gamma_a} \sqrt{q_a} = \text{fixed}, \quad g \equiv \frac{16\pi G_{10}}{V_1 V_4 L^2} = \text{fixed}. \quad (4.14)$$

The near-extremal limit of the three-charge solution (2.4)–(2.8) can now be written down. The metric and the dilaton are given by

$$ds^2 = \hat{H}_1^{-\frac{3}{4}} \hat{H}_4^{-\frac{3}{8}} \hat{H}_0^{-\frac{7}{8}} \left(-U dt^2 + \hat{H}_4 \hat{H}_0 dx^2 + \hat{H}_1 \hat{H}_0 \sum_{i=1}^4 (du^i)^2 + \hat{H}_1 \hat{H}_4 \hat{H}_0 V_{ab} dx^a dx^b \right), \quad (4.15)$$

$$e^{2\phi} = \hat{H}_1^{-1} \hat{H}_4^{-\frac{1}{2}} \hat{H}_0^{\frac{3}{2}}, \quad (4.16)$$

where from (4.13)

$$\hat{H}_a = \begin{cases} \hat{h}_a \frac{1-U}{c_t} & \text{for } \gamma_a > 0 \\ H_a & \text{for } \gamma_a = 0 \end{cases}, \quad \hat{h}_a \equiv \frac{(2\pi)^{1-2\gamma_a} \ell_a^2}{\Omega_2}. \quad (4.17)$$

In the case that all $\gamma_a > 0$ the dilaton is constant

$$e^{2\phi} = \hat{h}_1^{-1} \hat{h}_4^{-\frac{1}{2}} \hat{h}_0^{\frac{3}{2}}. \quad (4.18)$$

The non-vanishing components of the gauge fields are given by

$$(A_a)_{t\dots} = \begin{cases} \hat{H}_a^{-1} & \text{for } \gamma_a > 0, \\ \coth \alpha_a (H_a^{-1} - 1) & \text{for } \gamma_a = 0. \end{cases} \quad (4.19)$$

Note that the near-horizon limit of the extremal three-charge metric is $AdS_2 \times S^3 \times T^5$ for the localized phase of F1-D0-D4. For the uniformly smeared phase there is not such a simple description.

Relation to string scale units

Before discussing the physical quantities of the new near-extremal solution, it is useful to see how the parameters that are kept fixed in near-extremal limit, namely g and ℓ_a , are related to the string coupling g_s and the string length ℓ_s . By comparing the parameters \hat{h}_a in equation (4.17) to the usual harmonic functions of smeared extremal branes, we find

$$\ell_1^2 = L^{2\gamma_1} \frac{(2\pi\ell_s)^6 g_s^2 N_1}{L^2 V_4}, \quad (4.20)$$

$$\ell_4^2 = L^{2\gamma_4} \frac{(2\pi\ell_s)^3 g_s N_4}{L^2 V_1}, \quad (4.21)$$

$$\ell_0^2 = L^{2\gamma_0} \frac{(2\pi\ell_s)^7 g_s N_0}{L^2 V_1 V_4}, \quad (4.22)$$

and

$$g = \frac{(2\pi)^7 \ell_s^8 g_s^2}{L^2 V_1 V_4} \quad (4.23)$$

where N_1 is the number of F1-strings, N_4 is the number of D4-branes and N_0 is the number of D0-branes.

From the fixed parameters in equations (4.20)–(4.23) we can form the dimensionless combination

$$\frac{\ell_1 \ell_4 \ell_0}{g} = 2\pi \sqrt{N_1 N_4 N_0} \quad (4.24)$$

which will be useful to reinstate the units in the rescaled entropy that we obtain below.

4.2 Physical quantities

In the near-extremal limit we define the energy above extremality and the tensions in the compact directions as⁸

$$E = \lim_{L \rightarrow 0} \left(\bar{M} - \sum_a Q_a \right), \quad \hat{T}_z = \lim_{L \rightarrow 0} \frac{L}{2\pi} \bar{T}_z, \quad L_a \hat{T}_a = \lim_{L \rightarrow 0} (L_a \bar{T}_a - Q_a). \quad (4.25)$$

The general definitions for the energy and tensions in backgrounds that are not asymptotically flat can be found in [44] (see also [50]). In reference [11] it was actually shown that for one-charge solutions written in the ansatz (3.1) the general definition is equivalent to (4.25). We have checked that the same is true for the three-charge case.

The dimensionless versions of these quantities are defined as

$$\epsilon = gE, \quad r = \frac{2\pi \hat{T}_z}{E}, \quad r_a = \frac{L_a \hat{T}_a}{E}. \quad (4.26)$$

These variables are the possible independent physical parameters analogous to μ and n in the neutral case and $\bar{\mu}$, \bar{n} and \bar{n}_a in the non-extremal case.

Using equations (2.24) and (2.25) and the fact that

$$\lim_{\alpha \rightarrow \infty} (\sinh^2 \alpha - \sinh \alpha \cosh \alpha) = -\frac{1}{2} \quad (4.27)$$

we find the energy above extremality in terms of \hat{c}_t and \hat{c}_z as

$$E = \frac{\Omega_2}{2\pi g} \left(\frac{1}{2} \hat{c}_t - \hat{c}_z \right) \quad (4.28)$$

while the tension of the transverse circle and the tensions in the spatial world-volume directions are given by

$$2\pi \hat{T}_z = \frac{\Omega_2}{2\pi g} (\hat{c}_t - 2\hat{c}_z), \quad L_a \hat{T}_a = \frac{\Omega_2}{2\pi g} \left(\frac{1}{2} \hat{c}_t - \hat{c}_z \right). \quad (4.29)$$

Rewriting the dimensionless energy ϵ and the relative tensions, r and r_a , in terms of the seeding μ and n we get the remarkable result

$$\epsilon = \frac{1}{2} \mu n, \quad r = 2, \quad r_a = 1, \quad a = 1, 4, 0. \quad (4.30)$$

Note that the relative tensions are constant. That means that the tensions are proportional to the energy above extremality. This is a very special result that depends on the fact that we have exactly three charges and four spatial transverse dimensions. In section 4.3 we will see that r being a constant is a necessary condition in order for the localized five-dimensional black hole to have a finite non-vanishing entropy in the extremal limit. But first we look at the near-extremal thermodynamics.

⁸We will always assume the Q_a s to be positive.

The near-extremal temperature and entropy are given by

$$\hat{T} = \lim_{L \rightarrow 0} \bar{T}, \quad \hat{S} = \lim_{L \rightarrow 0} \bar{S} \quad (4.31)$$

where \bar{T} and \bar{S} are the non-extremal temperature and entropy. To get rescaled temperature and entropy we need a new length scale and it turns out to be useful to define

$$\ell \equiv \ell_1 \ell_4 \ell_0 \quad (4.32)$$

with ℓ_a given in equation (4.7). Note that ℓ has the dimension of length since the γ_a sum to one. Dimensionless versions of the temperature and entropy in the near-extremal limit can now be defined by⁹

$$\hat{\mathfrak{t}} = \ell \hat{T}, \quad \hat{\mathfrak{s}} = \frac{g}{\ell} \hat{S}. \quad (4.33)$$

These quantities can be related to the non-extremal temperature and entropy via

$$\bar{\mathfrak{t}} \sqrt{q_1 q_4 q_0} = L \bar{T} \sqrt{q_1 q_4 q_0} \rightarrow \ell \hat{T} = \hat{\mathfrak{t}}, \quad (4.34)$$

$$\bar{\mathfrak{s}} / \sqrt{q_1 q_4 q_0} = \frac{g}{L \sqrt{q_1 q_4 q_0}} \bar{S} \rightarrow \frac{g}{\ell} \hat{S} = \hat{\mathfrak{s}}, \quad (4.35)$$

and this implies the map

$$\hat{\mathfrak{t}} \hat{\mathfrak{s}} = \bar{\mathfrak{t}} \bar{\mathfrak{s}} = \mathfrak{t} \mathfrak{s}. \quad (4.36)$$

Given the temperature and entropy of the neutral seeding solution, we find the rescaled temperature and entropy of the near-extremal three-charge solution as

$$\hat{\mathfrak{t}} = \mathfrak{t}(\mathfrak{t}\mathfrak{s})^{3/2}, \quad \hat{\mathfrak{s}} = \mathfrak{s}(\mathfrak{t}\mathfrak{s})^{-3/2} \quad (4.37)$$

This can be derived from

$$\hat{\mathfrak{t}} = \bar{\mathfrak{t}} \sqrt{q_1 q_4 q_0} = \frac{\sqrt{q_1 q_4 q_0}}{\cosh \alpha_1 \cosh \alpha_4 \cosh \alpha_0} \mathfrak{t} \quad (4.38)$$

by noticing that from the neutral Smarr formula $\mathfrak{t}\mathfrak{s} = (2 - n)\mu/3$ and equation (2.37) we have

$$\lim_{\alpha_a \rightarrow \infty} \frac{\sqrt{q_a}}{\cosh \alpha_a} = \sqrt{\mathfrak{t}\mathfrak{s}}. \quad (4.39)$$

In the near-extremal three-charge case we do not have a Smarr relation in the traditional sense since the relative tensions are constant. However, we can write a Smarr relation in a ‘mixed’ notation where we use the relative tension n of the seeding solution

$$\hat{\mathfrak{t}} \hat{\mathfrak{s}} = \frac{2(2 - n)}{3n} \epsilon. \quad (4.40)$$

From this and the first law of thermodynamics we obtain

$$\frac{\delta \log \hat{\mathfrak{s}}}{\delta \log \hat{\mathfrak{t}}} = \frac{3n}{2(2 - n)} \quad (4.41)$$

⁹This is assuming that all the charges are non-zero. If one or two of the charges are zero then the corresponding ℓ_a should be left out of the definition of ℓ . We will come back to this in section 6.

so that given the curve $n(\epsilon)$ we can find the entire thermodynamics.

The Helmholtz free energy is

$$\hat{f} = \epsilon - \hat{t}\hat{s}, \quad \delta\hat{f} = -\hat{s}\delta\hat{t} \tag{4.42}$$

and using the Smarr relation (4.40) we can rewrite this for near-extremal black holes on a circle as

$$\hat{f} = \frac{5n-4}{3n}\epsilon. \tag{4.43}$$

This is the near-extremal free energy written in mixed notation, using the neutral tension n instead of r which is a constant in this case.

Note that the free energy is negative for $n \leq 4/5$. This is important for the dual field theory which is only thermodynamically stable if the free energy is negative. The region $n \leq 4/5$ contains all the usual phases with $SO(3)$ symmetry (which have $n \leq 1/2$), and also some of the Kaluza-Klein bubbles [14].

From the first law of thermodynamics we get

$$\frac{\delta \log \hat{f}}{\delta \log \hat{t}} = -\frac{\hat{s}\hat{t}}{\hat{f}} = \frac{4-2n}{4-5n}. \tag{4.44}$$

Given $n(\hat{t})$ we can integrate the above equation and get \hat{f} as a function of \hat{t} . Note that we again have to use the relative tension of the seeding solution.

The world-volume pressure is

$$\hat{p}_a = -r_a\epsilon = -\epsilon \tag{4.45}$$

where we used (4.30). This is not proportional to the free energy (4.43), contrary to the one-charge solutions [11] for which the world-volume pressure is always equal to minus the free energy.

4.3 Finite entropy from the first law of thermodynamics

In this section we try to understand why the relative tension is a constant for the near-extremal three-charge solution. Let us start with an ansatz for the energy above extremality in terms of the seeding μ and n

$$\epsilon = (a + bn)\mu \tag{4.46}$$

where a and b depend on the number of charges and the number of transverse dimensions. We can argue for this ansatz using only the expression for the gauge fields (2.6)–(2.8). Since μ and μn are linear combinations of the seeding c_t and c_z it is enough to show that the same is true for the energy above extremality. To see that, we write

$$\epsilon = \mu + \sum_a (\nu_a - 1)q_a \tag{4.47}$$

where the chemical potential $\nu_a = -A_a|_{\text{Horizon}}$ is independent of c_t and c_z . Since q_a is proportional to c_t we immediately see that ϵ is indeed a linear combination of c_t and c_z .

In the next section we will see that ϵ takes the form (4.46) for the one- and two-charge cases with non-zero a and b [cf. eq. (6.20) and (6.24) for the near-extremal map in the one- and two-charge case respectively] but for the three-charge case we have seen that $a = 0$ [cf. (4.30)]. We will ignore this knowledge for now, and first examine what the first law of thermodynamics implies.

From the non-extremal Smarr formula (2.49) and the map (4.36) we know that the product of the rescaled entropy and temperature is given by

$$\hat{\mathfrak{t}}\hat{\mathfrak{s}} = \frac{2-n}{3}\mu = \frac{2-n}{3(a+bn)}\epsilon \tag{4.48}$$

where in the last equation we used the ansatz (4.46). Plugging this into the first law of thermodynamics, $\delta\epsilon = \hat{\mathfrak{t}}\delta\hat{\mathfrak{s}}$, we therefore find

$$\frac{\delta \log \hat{\mathfrak{s}}}{\delta \log \epsilon} = \frac{3(a+bn)}{2-n}. \tag{4.49}$$

For small black holes in the localized phase, $n \rightarrow 0$ as $\epsilon \rightarrow 0$, since the tension should vanish in the extremal (BPS) limit. Therefore it follows that for small black holes close to extremality

$$\frac{\delta \log \hat{\mathfrak{s}}}{\delta \log \epsilon} \simeq \frac{3a}{2}. \tag{4.50}$$

Integrating this equation for small ϵ gives

$$\hat{\mathfrak{s}} \simeq A\epsilon^{3a/2} \tag{4.51}$$

where A is a constant of integration. But for this type of a localized black hole with three-charges in five spacetime dimensions, we expect to find [1]

$$\hat{\mathfrak{s}} \rightarrow \text{constant} \neq 0 \tag{4.52}$$

as $\epsilon \rightarrow 0$. This can only be true if $a = 0$. We have therefore seen that in order for the entropy of the small three-charge black hole to be non-vanishing in the extremal limit, the number a in the ansatz (4.46) should be zero. This is what makes the three-charge case special compared to the one- and two-charge case. The fact that a vanishes has an immediate consequence for the relative tension

$$r = \frac{L\bar{\mathcal{T}}_z}{E} = \frac{\mu n}{\epsilon} = \frac{n}{a+bn} = \frac{1}{b}. \tag{4.53}$$

From equation (4.30) we see that the five-dimensional near-extremal three-charge black hole indeed has $a = 0$ and $b = 1/2$ which gives the correct value $r = 2$.

Let us finally note that we can quickly see how r depends on the number of transverse dimensions and charges.¹⁰ Firstly, in this general case we still have that $L\bar{\mathcal{T}}_z = L\mathcal{T}_z = \mu n$ (see appendix C). Thus we only have to see how a and b depends on d , the number of

¹⁰We choose a short derivation here. One can also obtain the result by calculating \bar{c}_t, \bar{c}_z , etc.

transverse spatial dimensions, and N_{ch} , the number of charges. We assume that (2.32) applies and that the form of the gauge fields is the same in the general case such that $\nu_a = \tanh \alpha_a$ and

$$M^{\text{el}} = \sum_{a=1}^{N_{\text{ch}}} \tanh \alpha_a Q_a. \tag{4.54}$$

Further, from [11] we get $c_t \propto \frac{(d-2)M-LT}{(d-2)^2-1}$. Using this and the form of the gauge fields, we see that $Q_a = (d-3) \sinh \alpha_a \cosh \alpha_a \frac{(d-2)M-LT}{(d-2)^2-1}$ thus giving

$$\epsilon = \left(1 - \frac{N_{\text{ch}}(d-2)}{2(d-1)} \right) \mu + \frac{N_{\text{ch}}}{2(d-1)} \mu n \tag{4.55}$$

where we have used that $(\tanh \alpha_a - 1) \sinh \alpha_a \cosh \alpha_a \rightarrow -1/2$ as $\alpha_a \rightarrow \infty$. This, of course, agrees with our case where $N_{\text{ch}} = 3$ and $d = 4$. The only other case with $a = 0$ is for $N_{\text{ch}} = 4$ and $d = 3$. For the latter case it is actually known that one can have configurations with finite entropy (see e.g. [51]). However, our derivation does not hold in this case since the asymptotic c_t and c_z do not make sense for $d = 3$.

5. Phase diagrams for the near-extremal case

In this section we discuss consequences of the near-extremal map for the different phases of the seeding solution considered in section 3 and display the phase diagrams.

5.1 Energy versus relative tension

In normal situations it would be appropriate to draw the different phases of near-extremal solutions on an (ϵ, r) phase diagram. But in the special case of a five-dimensional three-charge black holes on a circle the relative tension r is a constant independent of the seeding solution. The phase diagram is therefore just a straight line $r(\epsilon) = 2$ which does not contain much information about the different phases.

We can, however, see how the relative tension approaches this constant as the charges are sent to infinity. In this discussion we define, with a slight abuse of notation, the *non-extremal* energy above extremality for finite charges as $\epsilon = \bar{\mu} - \sum_a q_a$. From the definition of the relative tension, we then have that

$$r = \frac{L\bar{T}_z}{E} = \frac{\mu n}{\epsilon}. \tag{5.1}$$

We can plug in our equations for the tension and energy above extremality (2.42) and get

$$\frac{\mu n}{\bar{\mu} - \sum_a q_a} = \left(\frac{1}{2} + \frac{(2-n)}{6n} \sum_a \frac{b_a}{1 + \sqrt{1 + b_a^2}} \right)^{-1}. \tag{5.2}$$

This quantity clearly goes to $r = 2$ in the near-extremal limit, since by equation (2.40) the b_a vanish when the charges go to infinity. We do not have analytic expressions for the

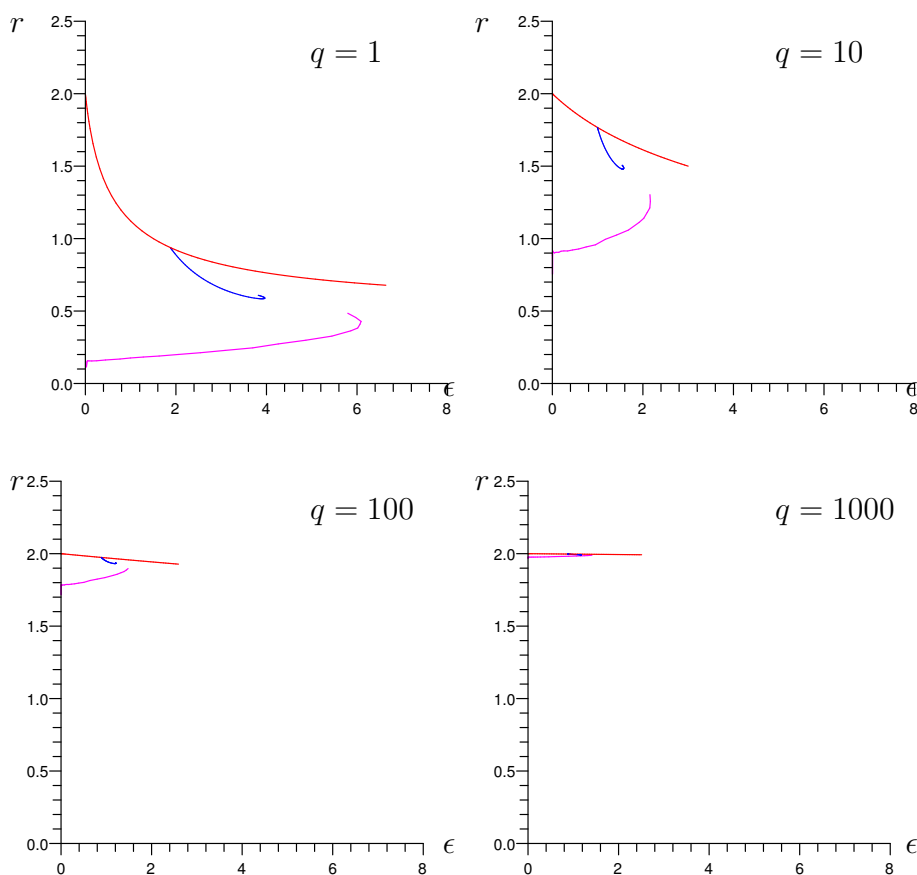


Figure 2: The non-extremal (ϵ, r) phase diagram for four different values of the charges. The three charges are all taken to be equal and have the value $q = 1$, $q = 10$, $q = 100$, and $q = 1000$. Notice how all the phases collapse to the line $r = 2$ as the charges go to infinity. The curves were found from equation (5.2) using $n = 1/2$ for the uniform phase (red curve), numerical data from [23] for the localized phase (magenta curve) and numerical data from [24] for the non-uniform phase (blue curve).

full localized phase nor for the non-uniform phase, but from the numerical data [23, 24] we can plot a non-extremal (ϵ, r) phase diagram and see how it evolves as the charges go to infinity. Figure 2 depicts this phase diagram for four increasing values of the charges and we clearly see how all the phases collapse to the degenerate line $r = 2$ as the charges go to infinity.

5.2 Thermodynamics of the uniform and non-uniform phases

The thermodynamics of the uniform phase in the near-extremal limit follows directly from the general map (4.30), (4.37) and the known thermodynamics $(\mathfrak{s}_u(\mu) = \mu^2/4)$ of the uniform black string in five dimension, and we find

$$\hat{\mathfrak{s}}_u(\epsilon) = \sqrt{2\epsilon}, \quad \hat{\mathfrak{f}}_u(\hat{t}) = -\frac{1}{2}\hat{t}^2. \tag{5.3}$$

If we apply the general map (4.30) to the neutral non-uniform branch that was reviewed in section 3.3 we get a new non-uniform phase of near-extremal three-charge black holes on a circle. The Gregory-Laflamme point $(\mu_{\text{GL}}, 1/2)$ where the non-uniform phase branches off the uniform phase is mapped to a critical point with energy above extremality $\epsilon_c = \mu_{\text{GL}}/4$. The relative tension at this point is $r = 2$ as for all other points and therefore we cannot describe the non-uniform phase as a curve on the (ϵ, r) diagram like in the non-extremal case. We can, however, express the neutral tension n in terms of $\epsilon - \epsilon_c$ near the critical point. The expression is

$$n(\epsilon) = \frac{1}{2} - \hat{\gamma}(\epsilon - \epsilon_c) + \mathcal{O}((\epsilon - \epsilon_c)^2) \tag{5.4}$$

with $\hat{\gamma}$ given by

$$\hat{\gamma} = \frac{4\gamma}{1 - 2\gamma\mu_{\text{GL}}} = 38.89 \tag{5.5}$$

where $\gamma = 0.14$ is the slope of the neutral non-uniform branch and $\mu_{\text{GL}} = 3.52$ is the Gregory-Laflamme critical mass. It is useful to have the neutral tension in terms of the energy above extremality because we can integrate (4.41) to find the entropy for the non-uniform branch to leading order

$$\hat{s}_{\text{nu}}(\epsilon) = \hat{s}_c \left(1 + \frac{\epsilon - \epsilon_c}{2\epsilon_c} - \left(\frac{1}{8} + \frac{2}{3}\hat{\gamma}\epsilon_c \right) \frac{(\epsilon - \epsilon_c)^2}{\epsilon_c^2} \right) + \mathcal{O}((\epsilon - \epsilon_c)^3) \tag{5.6}$$

where $\hat{s}_c = \sqrt{2\epsilon_c}$ is the critical entropy.

We can recover the entropy of the uniform branch by replacing $\hat{\gamma}$ with zero in the expression (5.6) above. Notice that the entropy of the non-uniform phase deviates from that of the uniform phase only to second order. These two phases¹¹ are depicted in figure 3 together with the localized phase which will be discussed in section 5.3.

In the canonical ensemble we can get the free energy of the non-uniform phase as an expansion around the critical temperature $\hat{t}_c = \sqrt{2\epsilon_c}$. Using the Smarr formula to relate temperature to energy above extremality, we get from (5.6)

$$\hat{f}_{\text{nu}} = -\epsilon_c - \hat{s}_c(\hat{t} - \hat{t}_c) - \frac{c}{2\hat{t}_c}(\hat{t} - \hat{t}_c)^2 + \mathcal{O}((\hat{t} - \hat{t}_c)^3) \tag{5.7}$$

where

$$c = \frac{3\hat{s}_c}{3 + 16\hat{\gamma}\epsilon_c} = 0.0072 \tag{5.8}$$

is the heat capacity of the non-uniform phase at $\hat{t} = \hat{t}_c$. The free energy of the uniform branch around $\hat{t} = \hat{t}_c$ is also given by (5.7) but with heat capacity $c = \hat{s}_c$ as can be easily derived from (5.3). These phases are depicted in figure 4 together with the localized branch which will be discussed in section 5.3.

¹¹Note that for each of these two phases we also have copies, which are mapped from the copies [48, 37, 11] of the non-uniform and localized phase of the seeding solution. The thermodynamic quantities of the copies of the near-extremal three-charge solutions are given by $\tilde{\epsilon} = \epsilon/k$, $\tilde{t} = \hat{t}/\sqrt{k}$, $\tilde{s} = \hat{s}/\sqrt{k}$ where $k = 2, 3, \dots$

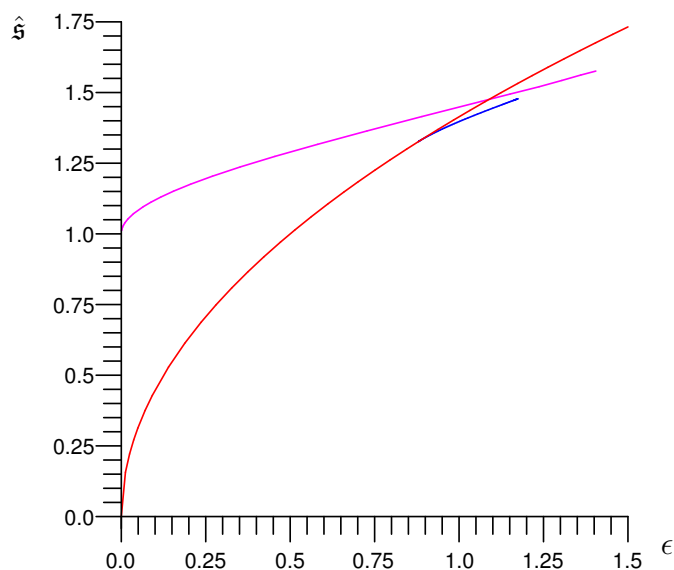


Figure 3: The entropy \hat{s} as a function of the energy above extremality ϵ for the localized phase (magenta), the uniform phase (red) and the non-uniform phase (blue). The curves are based on numerical data from [23, 24].

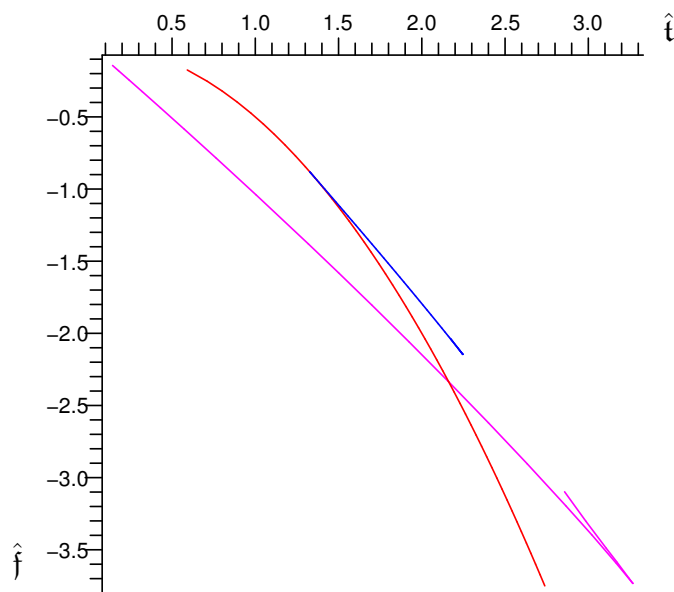


Figure 4: The free energy \hat{f} as a function of the temperature \hat{t} for the localized phase (magenta), the uniform phase (red) and the non-uniform phase (blue). The curves are based on numerical data from [23, 24].

5.3 Thermodynamics of small three-charge black holes on a circle

We now consider the case of a small localized black hole. In the neutral case we have (see

section 3.4)

$$\mu = \frac{3}{2}\rho_0^2 + \frac{1}{32}\rho_0^4 + \mathcal{O}(\rho_0^8), \quad n = \frac{1}{24}\rho_0^2 - \frac{1}{1152}\rho_0^4 + \mathcal{O}(\rho_0^6) \quad (5.9)$$

which gets mapped by (4.30) to the energy above extremality

$$\epsilon = \frac{1}{32}\rho_0^4 + \mathcal{O}(\rho_0^8). \quad (5.10)$$

Note that not only is the ρ_0^2 term missing but the ρ_0^6 term cancels as well. The rescaled entropy and temperature are by (4.37) mapped into

$$\hat{\mathfrak{s}} = 1 + \frac{\rho_0^2}{16} + \frac{\rho_0^4}{512} + \mathcal{O}(\rho_0^6), \quad (5.11)$$

$$\hat{\mathfrak{t}} = \rho_0^2 - \frac{\rho_0^4}{16} + \frac{\rho_0^6}{512} + \mathcal{O}(\rho_0^8). \quad (5.12)$$

Using (5.10) we can also write the entropy in terms of ϵ

$$\hat{\mathfrak{s}}_{\text{loc}}(\epsilon) = 1 + \sqrt{\frac{\epsilon}{8}} + \frac{\epsilon}{16} + \mathcal{O}(\epsilon^{3/2}). \quad (5.13)$$

showing the first two corrections to the extremal entropy for a thermal black hole localized on a circle. We correctly see that the entropy (5.13) goes to a non-vanishing constant in the extremal limit $\epsilon \rightarrow 0$. It is not surprising that the first correction comes with a power of ϵ smaller than one, otherwise the temperature would not go to zero in the extremal limit.

Restoring the normalization of the entropy using (4.24) we find

$$\hat{S}_{\text{loc}} = \frac{\ell \hat{\mathfrak{s}}}{g} = 2\pi \sqrt{N_1 N_4 N_0} \left(1 + \sqrt{\frac{\epsilon}{8}} + \frac{\epsilon}{16} + \mathcal{O}(\epsilon^{3/2}) \right). \quad (5.14)$$

The entropy in the extremal limit is the constant $\hat{S}_0 = 2\pi \sqrt{N_1 N_4 N_0}$ in agreement with the well-known result of [1]. Eq. (5.14) is one of the central results of the paper and gives, as a function of the energy above extremality, the first two corrections to the finite entropy due to the interactions of the black hole. We will come back to this in section 7, where we will present a microscopic counting of the corrected entropy in case of the partial extremal limit described in section 6.1.

The Helmholtz free energy of small localized black holes in the canonical ensemble (4.42) is given by

$$\hat{\mathfrak{f}}_{\text{loc}}(\hat{\mathfrak{t}}) = -\hat{\mathfrak{t}} - \frac{1}{32}\hat{\mathfrak{t}}^2 - \frac{1}{512}\hat{\mathfrak{t}}^3 + \mathcal{O}(\hat{\mathfrak{t}}^4). \quad (5.15)$$

The fact that the leading term in the free energy is linear in $\hat{\mathfrak{t}}$ is in accord with the localized black hole background being asymptotically $AdS_2 \times S^3 \times T^5$. One expects the dual gauge theory to be quantum mechanical and hence the free energy to be proportional to the temperature. The higher order terms are then due to the presence of the circle.

It is also interesting to see how the thermodynamics changes if only one or two of the charges are sent to infinity with the others kept finite. These cases are studied in the next section.

6. Other near-extremal limits

In this section we discuss some other limiting cases for the near-extremal three-charge background, involving one or two finite charges. We also present the special case of non-extremal solutions with two charges only and present the corresponding near-extremal limit.

6.1 Near-extremal limit with finite charges

We start by considering near-extremal limits where one or two of the three charges stay finite. This corresponds to having one or more of the γ_a in (4.14) vanishing.

One finite charge

Without loss of generality we can choose any of the three charges finite. We choose here to take $q_4, q_0 \rightarrow \infty$ with q_1 finite. The presence of this finite charge corresponds to choosing $\gamma_1 = 0$ with $\gamma_4, \gamma_0 > 0$ in the near-extremal limit (4.14). The explicit form of the resulting background is easily obtained using the general expressions in (4.15)–(4.19). This limit is also called the dilute gas limit in the literature [3] and after a T-duality in the x -direction where the F1-string lies, corresponds to the near-extremal D1-D5 brane system with finite KK momentum in the direction of the D-string.

In close analogy to (4.25), the energy and tensions in this partial limit are defined as

$$E = \lim_{L \rightarrow 0} \left(\bar{M} - \sum_{a=0,4} Q_a \right), \quad \hat{T}_z = \lim_{L \rightarrow 0} \frac{L}{2\pi} \bar{T}_z, \\ \hat{T}_1 = \lim_{L \rightarrow 0} (L_1 \bar{T}_1 - L_1 \bar{T}_1^{\text{el}}), \quad L_a \hat{T}_a = \lim_{L \rightarrow 0} (L_a \bar{T}_a - Q_a) \quad \text{for } a = 0, 4. \quad (6.1)$$

The dimensionless versions of these quantities are taken to be

$$\epsilon = gE, \quad r = \frac{2\pi \hat{T}_z}{E - M_1^{\text{el}}}, \quad r_a = \frac{L_a \hat{T}_a}{E - M_1^{\text{el}}}, \quad \hat{\mathfrak{t}} = \ell \hat{T}, \quad \hat{\mathfrak{s}} = \frac{g}{\ell} \hat{S} \quad (6.2)$$

where M_1^{el} is the electric mass defined in (2.31), g is defined in (2.15) and $\ell = \ell_0 \ell_4$.

Using the definitions (6.1), (6.2) and the results in (2.24)–(2.29) for the physical quantities of the general three-charge background, one finds after some algebra

$$\epsilon = \frac{1+n}{3} \mu + \mu_1^{\text{el}}, \quad r = \frac{3n}{1+n}, \quad r_1 = 1, \quad r_a = \frac{3n}{2(1+n)}, \quad \text{for } a = 4, 0, \quad (6.3)$$

where we used equations (2.23) to write the final result in terms of the physical parameters μ and n of the original seeding black hole. This provides for this partial near-extremal case the map from the neutral solution to the charged one. In this case, the only relative tension that is constant is the one in the spatial world-volume direction corresponding to the charge that is finite.

For temperature and entropy one easily finds the mapping

$$\hat{\mathfrak{t}} = \frac{\mathfrak{t}^2 \mathfrak{s}}{\cosh \alpha_1}, \quad \hat{\mathfrak{s}} = \mathfrak{t}^{-1} \cosh \alpha_1. \quad (6.4)$$

We also recall that $\mu_1^{\text{el}} = \nu_1 q_1$, with the chemical potential ν_1 and charge q_1 given by (2.33), (2.37) in terms of α_1 . Finally, the Smarr relation in this case takes the form

$$\hat{\mathbf{t}}\hat{\mathbf{s}} = (2 - r) \left(\epsilon - \mu_1^{\text{el}} \right). \quad (6.5)$$

The above map can of course be applied in particular to the neutral solutions that fall into the black hole/string ansatz (3.2), as was done in section 5 for the full near-extremal limit. For later use, we present here the result for the localized phase, obtained by applying the map (6.3) to the localized black hole on a circle. The corrected background for the near-extremal two-charge localized black hole follows by taking the near-extremal limit (4.14) of the non-extremal background (3.11) with the appropriate choice of γ_a . In particular, this amounts to $H_a \rightarrow \hat{H}_a$ where \hat{H}_a are given in (4.17).

The thermodynamic quantities of the resulting near-extremal localized phase, carrying one finite charge q_1 are then given by

$$\epsilon = \rho_0^2 \sinh^2 \alpha_1 + \frac{1}{2} \rho_0^2 \left(1 + \frac{1}{16} \rho_0^2 \right) + \mathcal{O}(\rho_0^6), \quad (6.6)$$

$$r = \frac{1}{8} \rho_0^2 - \frac{1}{128} \rho_0^4 + \mathcal{O}(\rho_0^6), \quad (6.7)$$

$$\hat{\mathbf{s}} = \rho_0 \cosh \alpha_1 \left(1 + \frac{1}{16} \rho_0^2 + \frac{1}{512} \rho_0^4 \right) + \mathcal{O}(\rho_0^7), \quad (6.8)$$

$$\hat{\mathbf{t}} = \frac{\rho_0}{\cosh \alpha_1} \left(1 - \frac{1}{16} \rho_0^2 + \frac{1}{512} \rho_0^4 \right) + \mathcal{O}(\rho_0^7). \quad (6.9)$$

In section 7 we will provide a microscopic derivation of the entropy found in the case of one finite charge. However, we will consider a permuted version of the above limit, namely with the D0-brane charge kept finite and F1 and D4-brane charge sent to infinity. It is not difficult to see that this case is completely analogous to the one discussed above.

In these expressions, we can send $\alpha_1 \rightarrow 0$ (and hence $q_1 \rightarrow 0$) and obtain the entropy and temperature of a small localized two-charge black hole on a circle (see section 6.2). The entropy clearly vanishes in the extremal limit $\rho_0 \rightarrow 0$.

Two finite charges

We now choose $q_4 \rightarrow \infty$ with q_1 and q_0 finite, corresponding to taking $\gamma_4 = 1$ with $\gamma_1 = \gamma_0 = 0$ in the expressions (4.14). Again, the explicit form of the resulting background is easily obtained using the general expressions in (4.15)–(4.19).

The energy and tensions in this partial limit are defined by the obvious generalizations of (6.1) and the dimensionless quantities are similar to those in (6.2), where we now divide by $E - M_1^{\text{el}} - M_0^{\text{el}}$.

We then find after some algebra the map

$$\epsilon = \frac{4+n}{6} \mu + \mu_1^{\text{el}} + \mu_0^{\text{el}}, \quad (6.10)$$

$$r = \frac{6n}{4+n}, \quad r_a = \frac{2(1+n)}{4+n}, \quad \text{for } a = 1, 0, \quad r_4 = \frac{3n}{4+n}, \quad (6.11)$$

where we recall that $\mu_a^{\text{el}} = \nu_a q_a$, with the chemical potential ν_a and charge q_a given by (2.33), (2.37) in terms of α_a . For vanishing q_1 and q_0 the above results agree with the one-charge $d = 4$ case considered in [11].

For temperature and entropy one easily finds the mapping

$$\hat{t} = \frac{t^{3/2} \mathfrak{s}^{1/2}}{\cosh \alpha_1 \cosh \alpha_0}, \quad \hat{s} = t^{-1/2} \mathfrak{s}^{1/2} \cosh \alpha_1 \cosh \alpha_0 \quad (6.12)$$

Finally, the Smarr relation in this case takes the form

$$\hat{t} \hat{s} = \frac{1}{2} (2 - r) \left(\epsilon - \mu_1^{\text{el}} - \mu_0^{\text{el}} \right). \quad (6.13)$$

As before, all of this can be applied to the ansatz, and in particular for the localized phase we now get

$$\epsilon = \rho_0^2 (\sinh^2 \alpha_1 + \sinh^2 \alpha_0) + \rho_0^2 \left(1 + \frac{1}{32} \rho_0^2 \right) + \mathcal{O}(\rho_0^6), \quad (6.14)$$

$$r = \frac{1}{16} \rho_0^2 - \frac{1}{512} \rho_0^4 + \mathcal{O}(\rho_0^6), \quad (6.15)$$

$$\hat{s} = \rho_0^2 \cosh \alpha_1 \cosh \alpha_0 \left(1 + \frac{1}{16} \rho_0^2 + \frac{1}{512} \rho_0^4 \right) + \mathcal{O}(\rho_0^8), \quad (6.16)$$

$$\hat{t} = \frac{1}{\cosh \alpha_1 \cosh \alpha_0} \left(1 - \frac{1}{16} \rho_0^2 + \frac{1}{512} \rho_0^4 \right) + \mathcal{O}(\rho_0^6). \quad (6.17)$$

The entropy vanishes in the extremal limit, as we expect, but one finds finite extremal temperature. This is in accord with the fact that we know that for $d = 4$ (see e.g. [11]) the localized phase corresponds to the near-extremal Type II NS5-brane, which has a Hagedorn temperature.

6.2 Two-charge black holes on a circle

Starting with the general three-charge non-extremal case one may also consider the situation with a smaller number of non-zero charges. In this section, we present some further details for the case with two non-zero charges, which was not studied before.

Brief review of one-charge case. As remarked earlier, when we set two of the three charges equal to zero, we should recover the one-charge case which was extensively studied in ref. [11, 40]. In particular, by sending say $q_1, q_0 \rightarrow 0$, eq. (2.42) becomes

$$\bar{\mu} - q_4 = \frac{(4+n)\mu}{6} + \frac{(2-n)\mu}{6} \frac{b_4}{1 + \sqrt{1 + b_4^2}} \quad (6.18)$$

which agrees with eq. (4.18) of [11] for $d = 4$. Recall that b_a was defined in equation (2.40). As an example, for the localized phase discussed in section 3.4 we can eliminate μ and n to arrive at [11]

$$\bar{n}(\bar{\mu}; q_4) = \frac{1}{24} (\bar{\mu} - q_4) + \mathcal{O}((\bar{\mu} - q_4)^2). \quad (6.19)$$

For comparison below, we also give here the map from the neutral five-dimensional Kaluza-Klein black holes to the near-extremal one-charge physical quantities

$$\epsilon = \frac{4+n}{6} \mu, \quad r = \frac{6n}{4+n}, \quad r_4 = \frac{3n}{4+n}, \quad \hat{t} = t^{3/2} \mathfrak{s}^{1/2}, \quad \hat{s} = t^{-1/2} \mathfrak{s}^{1/2}. \quad (6.20)$$

Non-extremal two-charge case. Turning to the two-charge case, we keep q_0, q_4 finite and take $q_1 \rightarrow 0$. The non-extremal background can simply be obtained by setting $\alpha_1 = 0$ in the general form in (2.4)–(2.9). For the thermodynamic quantities, we can use for example (2.42) to compute the non-extremal map

$$\bar{\mu} - \sum_{a=0,4} q_a = \frac{(1+n)\mu}{3} + \frac{(2-n)\mu}{6} \sum_{a=0,4} \frac{b_a}{1 + \sqrt{1 + b_a^2}} \quad (6.21)$$

and for the temperature and entropy we simply have (2.46), (2.47) with $\alpha_1 = 0$.

As an application of (6.21), it follows using the results of section 3.4 that for the localized phase

$$\bar{n}(\bar{\mu}; q_4, q_0) = \frac{1}{12} (\bar{\mu} - q_4 - q_0) + \mathcal{O}((\bar{\mu} - q_4 - q_0)^2). \quad (6.22)$$

We observe from (6.19) and (6.22) that in both the two-charge and the one-charge case, the relative tension for a small black hole is linear in the energy above extremality. For the three-charge case, which is special in many respects, this is not the case (see eq. (2.42)).

Near-extremal limit. We consider here the near-extremal limit of the two-charge black hole solution, in which we send both of the charges to infinity. Before discussing these results we briefly review the near-horizon limit of the corresponding extremal two-charge background.

For the localized phase, corresponding to D0-D4 smeared in the x -direction, we find after a T-duality in that direction the D1-D5 brane system, which has near-horizon geometry $AdS_3 \times S^3 \times T^4$. As a consequence we expect that the leading behavior of the thermodynamics of the localized phase in the near-extremal two-charge system with a transverse circle corresponds to that of a two-dimensional CFT. As we will see below this is indeed the case. For the uniform phase we find that the extremal background is described by a doubly-smeared configuration of D0-D4 branes, which after a double T-duality (in the x and z -direction) corresponds to the D2-D6 brane system. The dual description of this is less clear and presumably gravity is not decoupled, due to the presence of the D6-brane. However, as we will see below the system exhibits a Hagedorn behavior, in close analogy to the near-extremal NS5-brane system [52, 53].

The definition of the near-extremal two-charge limit follows the same route as discussed in section 4 for the three-charge case, where now the harmonic function H_1 is set to one. The corresponding background follows likewise from (4.15)–(4.17) by taking $\gamma_0 > 0, \gamma_4 > 0$ and setting $\hat{H}_1 = 1$. In this case the quantity ℓ (of dimension length) that enters the dimensionful physical quantities is given by

$$\ell = \ell_0 \ell_4 = \frac{(2\pi l_s)^5 g_s}{LV_1 \sqrt{V_4}} \sqrt{N_0 N_4} \quad (6.23)$$

where we used (4.21), (4.22). The quantity g is still given by (4.23).

The physical quantities are defined as in (4.25) with $\alpha_1 = 0$ (and hence $M_1^{\text{el}} = Q_1 = 0$) and the dimensionless quantities are as in (4.26) (in this case there is no \mathcal{T}_1). The map

follows easily by setting $q_1 = 0$ in the map (6.3), (6.4) so that the map from neutral Kaluza-Klein black holes to near-extremal two-charge physical quantities is

$$\epsilon = \frac{1+n}{3}\mu, \quad r = \frac{3n}{1+n}, \quad r_a = \frac{3n}{2(1+n)} \text{ for } a = 4, 0, \quad (6.24)$$

$$\hat{\mathbf{t}} = \mathbf{t}^2 \mathbf{s}, \quad \hat{\mathbf{s}} = \mathbf{t}^{-1}. \quad (6.25)$$

Like the one-charge map in (6.20), this is a one-to-one map that maps from the neutral two-dimensional (μ, n) phase diagram to the two-dimensional (ϵ, r) phase diagram of near-extremal two-charge solutions.

The Smarr relation is

$$\hat{\mathbf{t}}\hat{\mathbf{s}} = (2-r)\epsilon. \quad (6.26)$$

It is useful to recall that for a given curve in the (ϵ, r) phase diagram we can find the entire thermodynamics from this Smarr relation and the first law of thermodynamics $\delta\epsilon = \hat{\mathbf{t}}\delta\hat{\mathbf{s}}$, by integrating the equation

$$\frac{\delta \log \hat{\mathbf{s}}(\epsilon)}{\delta \log \epsilon} = \frac{1}{2-r(\epsilon)}. \quad (6.27)$$

In particular, for the solutions that are generated from the ansatz (3.1) the metric takes the form of (4.15)–(4.19) with $\hat{H}_1 = 1$. For the three known phases of black holes/strings on a cylinder discussed in section 3.2 we can then map to the corresponding phases of two-charge black holes with a circle in the transverse space.

For this we can use, as in section 3, the known data for the phases of five-dimensional Kaluza-Klein black holes along with the analytically known results for the uniform phase, the non-uniform phase near the GL point and localized phase in the small mass limit.¹²

The results can be summarized as follows. For the uniform phase we have

$$r_u(\epsilon) = 1, \quad \hat{\mathbf{s}}_u(\epsilon) = \epsilon, \quad \hat{\mathbf{f}}_u(\hat{\mathbf{t}}) = 0 \quad (6.28)$$

showing that this phase has Hagedorn thermodynamics with Hagedorn temperature $\hat{\mathbf{t}}_c = 1$ found e.g. from $t^{-1} = \partial s / \partial \epsilon$. For the non-uniform phase we have

$$r_{\text{nu}}(\epsilon) = 1 - \hat{\gamma} \cdot (\epsilon - \epsilon_c) + \mathcal{O}((\epsilon - \epsilon_c)^2), \quad (6.29)$$

$$\hat{\mathbf{s}}_{\text{nu}}(\epsilon) = \hat{\mathbf{s}}_u(\epsilon) \left(1 - \frac{\hat{\gamma}}{2\epsilon_c} (\epsilon - \epsilon_c)^2 + \mathcal{O}((\epsilon - \epsilon_c)^3) \right), \quad (6.30)$$

$$\hat{\mathbf{f}}_{\text{nu}}(\hat{\mathbf{t}}) = -\epsilon_c(\hat{\mathbf{t}} - 1) - \frac{1}{2\hat{\gamma}}(\hat{\mathbf{t}} - 1)^2 + \mathcal{O}((\hat{\mathbf{t}} - 1)^3) \quad (6.31)$$

$$\epsilon_c = \frac{\mu_{\text{GL}}}{2} = 1.76, \quad \hat{\gamma} = \frac{8\gamma}{3 - 2\mu_{\text{GL}}\gamma} = 0.56, \quad (6.32)$$

¹²Note that the non-uniform and localized phases also have copies, which are mapped from the copies [48, 37, 11] of the non-uniform and localized phase of the seeding solution. The thermodynamic quantities of the copies of the near-extremal two-charge solutions are given by $\tilde{\epsilon} = \epsilon/k$, $\tilde{\mathbf{t}} = \hat{\mathbf{t}}$, $\tilde{\mathbf{s}} = \hat{\mathbf{s}}/k$ where $k = 2, 3, \dots$

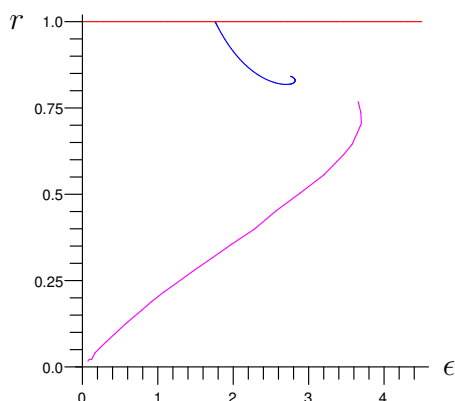


Figure 5: (ϵ, r) phase diagram for near-extremal two-charge black holes on a circle. Shown are the localized phase (magenta), the uniform phase (red) and the non-uniform phase (blue). The curves are based on numerical data from [23, 24].

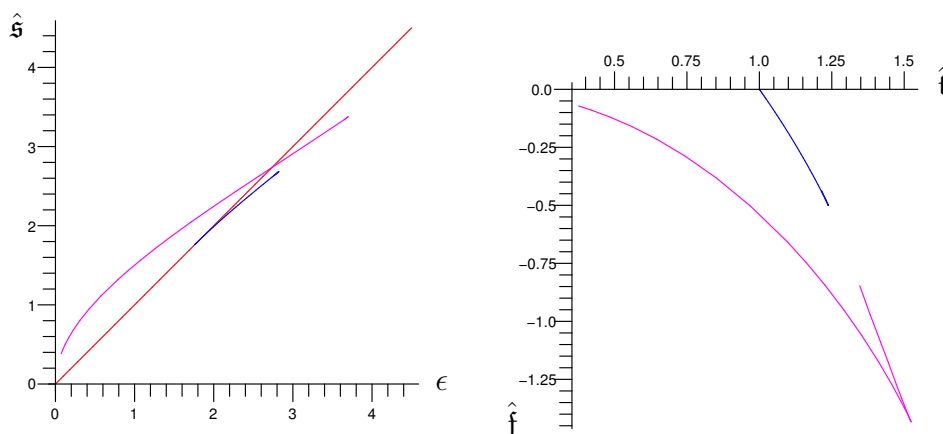


Figure 6: (ϵ, \hat{s}) and (\hat{t}, \hat{f}) diagrams for near-extremal two-charge black holes on a circle. Shown are the localized phase (magenta), the uniform phase (red) and the non-uniform phase (blue). The curves are based on numerical data from [23, 24].

exhibiting the departure at the critical point from the Hagedorn thermodynamics. Finally, for the localized phase

$$r_{\text{loc}}(\epsilon) = \frac{1}{4}\epsilon - \frac{1}{16}\epsilon^2 + \mathcal{O}(\epsilon^3), \tag{6.33}$$

$$\hat{s}_{\text{loc}}(\epsilon) = \sqrt{2}\epsilon^{1/2} \left(1 + \frac{1}{16}\epsilon - \frac{1}{512}\epsilon^2 + \mathcal{O}(\epsilon^3) \right), \tag{6.34}$$

$$\hat{f}_{\text{loc}}(\hat{t}) = -\frac{1}{2}\hat{t}^2 - \frac{1}{32}\hat{t}^4 - \frac{1}{256}\hat{t}^6 + \mathcal{O}(\hat{t}^8). \tag{6.35}$$

The numerically obtained plots for all of these quantities are shown in figures 5 and 6.

We first observe that the leading order thermodynamics (for small temperatures) of the localized phase of the near-extremal D1-D5 brane system on a transverse circle correctly

exhibits a free energy that is proportional to \hat{t}^2 , as expected for a two-dimensional conformal field theory at finite temperature. The results in (6.34), (6.35) then describe the departure of this behavior due to higher order temperature corrections in the presence of the circle.

On the other hand, note that the near-extremal uniformly smeared D1-D5 brane phase exhibits a Hagedorn temperature $T_{\text{hg}} = 1/\ell$ (since $\hat{t}_c = 1$) with ℓ given in (6.23), with the non-uniformly smeared phase emerging at the Hagedorn temperature. The picture that emerges in this system is in many respects analogous to the one considered in ref. [41] where the thermodynamics of near-extremal NS5-branes was studied, and applied to Little String Theory. In particular, we see from the above that also here, the localized phase provides a new stable phase in the canonical ensemble, extending to a maximum temperature that lies above the Hagedorn temperature. It would be interesting to see if there is a dual interpretation of this, which would require a further examination of the near-extremal D2-D6 brane system.

7. Microscopic entropy

In this section we use the microstate counting technique of [1, 5] to recalculate the entropy of our three-charge black holes on a circle. The non-extremal branes will interact across the transverse circle and this interaction effectively shifts the number of branes [10]. In the case of a small localized near-extremal black hole with one finite charge, we find agreement between the first correction obtained via microstate counting and the macroscopic corrected entropy in equation (6.6).

7.1 Review of non-extremal black hole microstate counting

Horowitz, Maldacena and Strominger [5] showed how to count the microstates for a special class of five-dimensional non-extremal black holes with three charges. In their case the metric asymptotes to Minkowski space with no circle in the transverse space.

In the weak string coupling limit the extremal black hole can be described as a configuration with N_4 D4-branes, N_1 fundamental strings and N_0 D0-branes. Non-extremal black holes can be generated, for example, by adding a small number $N_{\bar{0}}$ of anti-D0-branes. In a thermal system one would of course expect the non-extremality also to excite the D4-branes and the F1-strings, but we assume that the anti-D0-brane excitation is much lighter than the other two. The mass of the black hole is then given by the sum of the masses of each type of object

$$M = V_1\tau_1N_1 + V_4\tau_4N_4 + V_0\tau_0(N_0 + N_{\bar{0}}) \tag{7.1}$$

where V_a is the world-volume of each object, $\tau_1 = (2\pi\ell_s^2)^{-1}$ is the tension of the string, $\tau_4 = (g_s(2\pi)^4\ell_s^5)^{-1}$ is the tension of the D4-brane and $\tau_0 = (g_s\ell_s)^{-1}$ is the tension of the D0-brane. The charge associated to the D0-branes is given by

$$Q_0 = V_0\tau_0(N_0 - N_{\bar{0}}) \tag{7.2}$$

while the other charges are extremal and therefore given by $Q_a = V_a\tau_aN_a$ for $a = 1, 4$.

The D4-branes are separated in the spatial world-volume direction of the F1-string. There are therefore effectively $N_4 N_1$ strings between neighboring D4-branes. The D0-branes are like beads threaded on any one of these strings and in the dilute gas limit the strings are far apart and the beads can therefore only be threaded by one string at a time. In the extremal case with $N_{\bar{0}} = 0$, this gives rise to an entropy [5]

$$S = 2\pi\sqrt{N_1 N_4 N_0}. \tag{7.3}$$

In the non-extremal case the anti-D0-branes are like beads with opposite charge. In the dilute gas limit the forces between the beads are small and so interactions can be ignored. The entropy is additive in this case and given by [5]

$$S = 2\pi\sqrt{N_1 N_4} \left(\sqrt{N_0} + \sqrt{N_{\bar{0}}} \right). \tag{7.4}$$

This is the equation that we want to generalize for our near-extremal three-charge black hole with one finite charge on a circle.

7.2 Microstate counting on a circle

In the previous subsection the beads were far apart and did not interact, but with the small transverse circle present that is no longer a safe assumption to make. In our near-extremal limit, the size of the transverse circle was taken to be at the same scale as the energy above extremality and the interaction energy is therefore not negligible compared to the excitation energy. That means that interactions across the transverse circle between beads of opposite charge must be taken into account. The effect of the interaction is to shift the number of beads for a given total energy [10]. We now examine this for the localized phase of three-charge black holes on a circle.¹³

The non-extremal mass of the three-charge black hole (2.24) can be rewritten as

$$\bar{M} = \frac{\Omega_2 c_t}{2gL} \left[\left(1 - 2\frac{c_z}{c_t} \right) + \cosh 2\alpha_1 + \cosh 2\alpha_4 + \cosh 2\alpha_0 \right]. \tag{7.5}$$

The first term is equal to

$$\tilde{E} \equiv \frac{\Omega_2 c_t}{2gL} \left(1 - 2\frac{c_z}{c_t} \right) \tag{7.6}$$

and we notice that \tilde{E} is proportional to the tension along the transverse circle. This term is absent in the case where there is no transverse circle. The terms involving $\cosh 2\alpha_a$ are recognized as the contribution of each type of extended object to the total mass of five-dimensional three-charge black hole without the circle [5]. It is therefore natural to write

$$\bar{M} = \tilde{E} + \bar{M}_1 + \bar{M}_4 + \bar{M}_0 \tag{7.7}$$

¹³We thank Roberto Emparan for suggesting this computation to us.

with

$$\bar{M}_a = \frac{\Omega_2 c_t}{2gL} \cosh 2\alpha_a. \quad (7.8)$$

The charges (4.6) can also be rewritten as

$$Q_a = \frac{\Omega_2 c_t}{2gL} \sinh 2\alpha_a. \quad (7.9)$$

It is now easy to see that in the full near-extremal limit we have $\bar{M}_a - Q_a \rightarrow 0$ and $\bar{M} - \sum_a Q_a \rightarrow \tilde{E}$.

Partial extremal limit. Let us now consider the case where two of the charges are taken to be extremal, say Q_1 and Q_4 , while Q_0 has some small non-extremality. The total mass of the black hole is then

$$\bar{M} = Q_1 + Q_4 + \bar{M}_0 + \tilde{E}. \quad (7.10)$$

Following Costa and Perry [10],¹⁴ we want to write the total mass in the form

$$\bar{M} = Q_1 + Q_4 + \delta E + V_{\text{int}} \quad (7.11)$$

where δE is the energy carried by the D0-branes and the anti-D0-branes, and V_{int} is the interaction energy related to the presence of the transverse circle. As we start adding anti-D0-branes to the extremal system, they will interact with the D0-branes across the circle, reducing their energy by V_{int} .

The force between the beads across the circle gives rise to the tension \mathcal{T} and the interaction energy is the “energy stored in the tension”. In $d = 4$ the tension is proportional¹⁵ to L and therefore

$$V_{\text{int}} = - \int \mathcal{T} dL = -\frac{1}{2} \mathcal{T} L. \quad (7.12)$$

Notice that from the near-extremal map (4.30) we have $\frac{1}{2} \mathcal{T} L = \tilde{E}$ and hence $V_{\text{int}} = -\tilde{E}$.

Equation (7.10) can now be written as

$$\bar{M} = Q_1 + Q_4 + (\bar{M}_0 + 2\tilde{E}) + V_{\text{int}} \quad (7.13)$$

so we can identify

$$\delta E = \bar{M}_0 + 2\tilde{E}. \quad (7.14)$$

We find the effective number of D0- and anti-D0-branes from requiring

$$\delta E = V_0 \tau_0 (N'_0 + N'_0), \quad (7.15)$$

$$Q_0 = V_0 \tau_0 (N'_0 - N'_0). \quad (7.16)$$

¹⁴The same idea has been applied in [54].

¹⁵Since $\mathcal{T} = nM/L$ and $M \propto L^{d-2}$ [28] we have $\mathcal{T} \propto L^{d-3}$.

This gives

$$\tau_0 N'_0 = \frac{1}{2} (\bar{M}_0 + Q_0) + \tilde{E} = \frac{1}{2} \frac{\Omega_2 c_t}{2gL} \exp(2\alpha_0) + \tilde{E}, \quad (7.17)$$

$$\tau_0 N'_0 = \frac{1}{2} (\bar{M}_0 - Q_0) + \tilde{E} = \frac{1}{2} \frac{\Omega_2 c_t}{2gL} \exp(-2\alpha_0) + \tilde{E} \quad (7.18)$$

where we used the expression (7.9) for Q_0 . We thus see that there is a shift of \tilde{E} in the effective number of zero-branes compared to the black hole without the transverse circle [5, 10]. Recall that $\tau_0 = 1/g_s \ell_s$ and $V_0 = 1$.

The microstate entropy for our interacting system on a circle is then given by

$$S = 2\pi \sqrt{N_1 N_4} \left(\sqrt{N'_0} + \sqrt{N''_0} \right) \quad (7.19)$$

where N'_0 and N''_0 are the effective number of D0- and anti-D0-branes given in equations (7.17)–(7.18).

Application to small localized three-charge black holes. For small (neutral) localized black holes we have $c_t = \rho_0^2 L / (4\pi)$ and recall from eq. (3.14) that $c_z / c_t = \chi = 1/2 - \rho_0^2 / 32 + \mathcal{O}(\rho_0^6)$. We therefore get that

$$\frac{\Omega_2 c_t}{2L} = \frac{1}{2} \rho_0^2, \quad g\tilde{E} = \frac{1}{32} \rho_0^4 + \mathcal{O}(\rho_0^6) \quad (7.20)$$

where we used the definition of \tilde{E} in (7.6). We thus compute from (7.17), (7.18) the expressions

$$g\tau_0 N'_0 = \frac{1}{4} \rho_0^2 \exp(2\alpha_0) + \frac{1}{32} \rho_0^4 + \mathcal{O}(\rho_0^6), \quad (7.21)$$

$$g\tau_0 N''_0 = \frac{1}{4} \rho_0^2 \exp(-2\alpha_0) + \frac{1}{32} \rho_0^4 + \mathcal{O}(\rho_0^6). \quad (7.22)$$

To this order we therefore get

$$\sqrt{N'_0} + \sqrt{N''_0} = \frac{\sqrt{N_0}}{\ell_0} \rho_0 \cosh \alpha_0 \left(1 + \frac{\rho_0^2}{16} + \mathcal{O}(\rho_0^4) \right) \quad (7.23)$$

where we have used equation (4.22) to rewrite $g\tau_0 = \ell_0^2 / N_0$. The microstate entropy in equation (7.19) then becomes

$$S = \frac{2\pi \sqrt{N_1 N_4 N_0}}{\ell_0} \rho_0 \cosh \alpha_0 \left(1 + \frac{\rho_0^2}{16} + \mathcal{O}(\rho_0^4) \right) \quad (7.24)$$

$$= \frac{\ell_1 \ell_4}{g} \rho_0 \cosh \alpha_0 \left(1 + \frac{\rho_0^2}{16} + \mathcal{O}(\rho_0^4) \right). \quad (7.25)$$

This agrees with our previous result for the partial near-extremal entropy obtained from the black hole side (6.6), up to the order ρ_0^4 term.

We could in principle include higher order terms in equations (7.20) and hope to find agreement in the entropy to higher order, but it is not clear to which degree the method

of shifting the effective number of branes is accurate. We do not expect the microstate picture to hold for the uniform or the non-uniform phase so it is clear that somewhere on the way it must break down.

From equation (5.10) we know that the ρ_0^6 order term in \tilde{E} is vanishing and this information would yield $(1 - 2 \cosh \alpha_0)\rho_0^4/512$ as the next order term in the parenthesis of equation (7.24). This is to be compared to $\rho_0^4/512$ from the Bekenstein-Hawking entropy on the black hole side. It is not too surprising to find a minor discrepancy at such a high order.

8. Conclusions

In this paper we have generated new three-charge black hole solutions for the situation in which the three-charge black holes have a transverse circle. As we have shown, this case is interesting since the presence of the circle has a non-trivial effect for the physics of the three-charge system, giving rise to several new phenomena. Of particular interest is the case of a small three-charge black hole localized on the circle. Here we are deformed away from the non-zero entropy $S = 2\pi\sqrt{N_1 N_4 N_0}$ of the extremal case.

The starting point of our work is the construction of a map from neutral black hole solutions in five-dimensional Kaluza-Klein space-time to the three-charge black holes. This map is obtained using boosts and U-dualities, and is an extension of the map for the one-charge case obtained in ref. [11]. We restricted ourselves to solutions without Kaluza-Klein bubbles, and hence the phases we found were the localized phase, where the three-charge black is localized on the circle, the uniform phase, where the three-charge black hole is smeared uniformly around the circle, and the non-uniform phase, where the three-charge black hole is smeared without gaps on the circle in a translationally non-invariant fashion. As a consequence of the mapping of phases we also find an ansatz for three-charge black holes on a circle which applies to all of the considered new phases.

After constructing the non-extremal solution we turned to the near-extremal limit of the solutions. Here we have several new phenomena appearing that make the three-charge case special in comparison to the one-charge case of [11]. One of our results is that the relative tension always is constant $r = 2$, i.e. the tension along the circle is always proportional to the energy with the same constant of proportionality, even for the localized case when the energy is very small. We show that this is not a coincidence but rather a hitherto unknown consequence of the non-zero entropy of the extremal limit.

Having constructed the three-charge solutions, we examined their physical properties, including the global thermodynamic stability in both the microcanonical and canonical ensembles. We find the corrections to the extremal black hole entropy for the case of the small three-charge black hole localized on a circle. We also obtain the corrections to the thermodynamics for the non-uniform phase when it is close to the critical point. Finally, we examine how the constant relative tension $r = 2$ is approached as the near-extremal limit $q \rightarrow \infty$ is taken.

We considered furthermore the two-charge solutions. One interesting feature in this case is that the system exhibits a Hagedorn behavior, with the localized phase providing a stable phase that extends up to a limiting temperature that lies above the Hagedorn

temperature. The same behavior was recently found in the thermodynamics of the near-extremal NS5-brane [41]. More generally, the two-charge system has the interesting aspect that the AdS/CFT correspondence tells us that our deformations of the usual two-charge solution without the presence of the transverse circle should have a counterpart as deformations of the dual two-dimensional CFT. This would be very interesting to investigate.

Finally, we examined whether one can extend the microstate counting of the entropy performed in [5] to the case of a small three-charge black hole localized on the transverse circle. This was done following a similar analysis as that of ref. [10]. Interestingly, we find that in fact the first correction to the entropy due to the presence of the transverse circle is in perfect agreement with the microstate counting of the entropy. This first correction takes into account the self-interaction of the black hole across the circle, giving rise to a non-zero potential depending on the circumference of the circle. It would be very interesting to see if this matching of the black entropy and microstate counting could work to higher order. However, this would seemingly require a better understanding of the contributions to the potential for the black hole, since at higher order effects like gravitational backreaction can play a role.

There are several future directions for research on three-charge black hole solutions with a transverse circle. One is to construct more new solutions for three-charge black holes. Indeed, using the map with the five-dimensional bubble-black hole sequences of [35] as neutral seeding solutions will produce new three-charge solutions with regular event horizons. This could be very interesting to examine, for one thing the free energy of the near-extremal limit of such solutions can be negative, as one can see from (4.43) and the fact that there exist plenty of bubble-black hole configurations with $n \leq 4/5$. This could very well hint at the existence of new stable phases of the three-charge system.

Another interesting avenue to pursue would be to consider in more detail the non-uniform phase and the stability of the uniform phase. For both the two-charge and three-charge black holes we found a new non-uniform phase emerging from the uniform phase where the black hole is smeared along the transverse circle. That we in this way can dress the non-uniform phase with charges has in the one-charge case been shown [13, 49] to be connected with a map from the Gregory-Laflamme mode of the neutral black string to an unstable mode of the singly-charged uniform phase. It would be interesting to examine whether this works similarly for the two- and three-charge black holes. This involves construction a map of the unstable mode which in addition to boosts and U-dualities also would include complex rotations as in [13, 49]. Obviously this would be highly interesting in view of the Correlated Stability Conjecture (CSC) of [55–58], which is examined in [49] for the one-charge case, also in view of recent work on the CSC for other brane bound states [59–61].

Since we successfully can match the black hole entropy and the microstate counting for a small three-charge black hole localized on a circle, it would be interesting to examine further the consequences of this for Mathur’s fuzzball proposal [6, 7]. Mathur’s proposal is to reproduce the black hole entropy by counting smooth supergravity solutions without an event horizon. It would be interesting if one could reproduce our result for the entropy in this way.

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A. The boosts and U-dualities

In this appendix we show how the neutral solution is charged up via boosts and U-dualities. Let us start with a static and neutral five-dimensional Kaluza-Klein black hole as a seeding solution. The metric of such a solution can be written in the form

$$ds_5^2 = -U dt^2 + \frac{L^2}{(2\pi)^2} V_{ab} dx^a dx^b \tag{A.1}$$

where $V_{ab} dx^a dx^b$ describes a cylinder in the asymptotic region of circumference L . There is no dilaton and no gauge fields. By adding flat dimensions x and u_i , $i = 1, \dots, 4$, and performing a series of boosts and U-dualities we can construct the ten-dimensional solution of Type IIA Supergravity given in section 2.2.

A.1 The route

Before going through the calculation, let us first schematically sketch the route that we will take. First we make a boost in t and x direction with boost-parameter α_1 :

	t	x	u_1	u_2	u_3	u_4	
(α_1) P	\times	\times					Type IIB

T-dualize in x direction:

	t	x	u_1	u_2	u_3	u_4	
(α_1) F1	\times	\times					Type IIA

Boost in t and x direction with boost-parameter α_4 :

	t	x	u_1	u_2	u_3	u_4	
(α_1) F1	\times	\times					
(α_4) P	\times	\times					Type IIA

Lift to M-theory by adding the 11th dimension y :

	t	y	x	u_1	u_2	u_3	u_4	
(α_1) M2	\times	\times	\times					
(α_4) P	\times		\times					M-theory

Go back to Type IIA by reducing on x :

	t	y	u_1	u_2	u_3	u_4	
(α_1) F1	\times	\times					Type IIA
(α_4) D0	\times						

T-dualize in u_1, u_2, u_3, u_4 :

	t	y	u_1	u_2	u_3	u_4	
(α_1) F1	\times	\times					Type IIA
(α_4) D4	\times		\times	\times	\times	\times	

Lift to M-theory again by adding an 11th dimension x :

	t	x	y	u_1	u_2	u_3	u_4	
(α_1) M2	\times	\times	\times					M-theory
(α_4) M5	\times	\times		\times	\times	\times	\times	

Boost in t and x with boost parameter α_0 :

	t	x	y	u_1	u_2	u_3	u_4	
(α_1) M2	\times	\times	\times					M-theory
(α_4) M5	\times	\times		\times	\times	\times	\times	
(α_0) W	\times	\times						

Go back to IIA by reducing on x :

	t	y	u_1	u_2	u_3	u_4	
(α_1) F1	\times	\times					Type IIA
(α_4) D4	\times		\times	\times	\times	\times	
(α_0) D0	\times						

We stop here with a configuration that is a thermal excitation of an F1-string, D4-brane and a D0-brane, but we could T-dualize in directions u_3 and u_4 and lift to M-theory once more to get the configuration:

	t	x	y	u_1	u_2	u_3	u_4	
(α_1) M2	\times	\times	\times					M-theory
(α_4) M2	\times			\times	\times			
(α_0) M2	\times					\times	\times	

This is a thermal excitation of a configuration that is known to be 1/8-BPS.

A.2 Transformation of the solution

We now examine how the solution transforms under the boosts and U-dualities described in the previous subsection. We start with the metric

$$ds_{10}^2 = -U dt^2 + dx^2 + \sum_{i=1}^4 (du^i)^2 + ds_4^2 \tag{A.2}$$

where we have introduced the shorthand $ds_4^2 \equiv \frac{L^2}{(2\pi)^2} V_{ab} dx^a dx^b$. This is considered to be a solution of Type IIB Supergravity with vanishing dilaton and no gauge fields present.

Under a Lorentz-boost along the x -axis with rapidity α_1 , the coordinates transform as

$$\begin{pmatrix} t_{\text{new}} \\ x_{\text{new}} \end{pmatrix} = \begin{pmatrix} \cosh \alpha_1 & \sinh \alpha_1 \\ \sinh \alpha_1 & \cosh \alpha_1 \end{pmatrix} \begin{pmatrix} t_{\text{old}} \\ x_{\text{old}} \end{pmatrix} \quad (\text{A.3})$$

and the metric becomes

$$\begin{aligned} ds_{10}^2 = & (-U \cosh^2 \alpha_1 + \sinh^2 \alpha_1) dt^2 - 2(1-U) \cosh \alpha_1 \sinh \alpha_1 dt dx \\ & + (-U \sinh^2 \alpha_1 + \cosh^2 \alpha_1) dx^2 + \sum_{i=1}^4 (du^i)^2 + ds_4^2. \end{aligned} \quad (\text{A.4})$$

There is an isometry in the x direction and we can therefore use equations (2.54) in [62] to T-dualize in that direction and get a solution of Type IIA Supergravity. The dilaton becomes (fields with/without a tilde are new/old)

$$e^{2\tilde{\phi}} = \frac{e^{\phi}}{g_{xx}} = \frac{1}{(-U \sinh^2 \alpha_1 + \cosh^2 \alpha_1)} = H_1^{-1} \quad (\text{A.5})$$

where we have defined

$$H_1 \equiv (-U \sinh^2 \alpha_1 + \cosh^2 \alpha_1) = 1 + (1-U) \sinh^2 \alpha_1. \quad (\text{A.6})$$

The components of the metric that change under the duality are

$$\tilde{g}_{xx} = \frac{1}{g_{xx}} = H_1^{-1} \quad (\text{A.7})$$

$$\tilde{g}_{tx} = 0 \quad (\text{A.8})$$

$$\tilde{g}_{tt} = g_{tt} - (g_{tx})^2/g_{xx} = -UH_1^{-1} \quad (\text{A.9})$$

and we get a Kalb-Ramond field with component

$$\tilde{B}_{tx} = \frac{g_{tx}}{g_{xx}} = \coth \alpha_1 (H_1^{-1} - 1). \quad (\text{A.10})$$

Therefore the solution has become

$$ds_{10}^2 = H_1^{-1} (-U dt^2 + dx^2) + \sum_{i=1}^4 (du^i)^2 + ds_4^2 \quad (\text{A.11})$$

$$e^{2\tilde{\phi}} = H_1^{-1} \quad (\text{A.12})$$

$$B = \coth \alpha_1 (H_1^{-1} - 1) dt \wedge dx \quad (\text{A.13})$$

and we see that we have already picked up one charge. Two to go.

To produce the second charge we make another Lorentz boost in the x direction, now with boost parameter α_4 . The effect on the metric is analogous to the one above, except

now all terms with dt and dx are multiplied with H_1^{-1} . The dilaton is a scalar and does therefore not transform and it turns out that B is also invariant because

$$dt_{\text{new}} \wedge dx_{\text{new}} = dt_{\text{old}} \wedge dx_{\text{old}}. \quad (\text{A.14})$$

We lift the boosted solution to M-theory by adding an eleventh dimension y in the following S-duality fashion

$$ds_{11}^2 = e^{-2\phi/3} ds_{10}^2 + e^{4\phi/3} (dy + A_\mu dx^\mu)^2. \quad (\text{A.15})$$

There is no gauge field A_μ in our solution and we therefore have, using $e^{2\phi} = H_1^{-1}$,

$$ds_{11}^2 = H_1^{1/3} ds_{10}^2 + H_1^{-2/3} dy^2. \quad (\text{A.16})$$

The B field gives rise to a three-form with non-vanishing components

$$C_{txy} = \coth \alpha_1 (H_1^{-1} - 1). \quad (\text{A.17})$$

There is no dilaton in 11 dimensions and this is therefore the full solution.

Let us rewrite the boosted part of the metric (the dt and dx components) before reducing on x . Defining $H_4 \equiv 1 + (1 - U) \sinh^2 \alpha_4$, we find

$$\begin{aligned} & (-U \cosh^2 \alpha_4 + \sinh^2 \alpha_4) dt^2 - 2(1 - U) \cosh \alpha_4 \sinh \alpha_4 dt dx + (-U \sinh^2 \alpha_4 + \cosh^2 \alpha_4) dx^2 \\ &= -H_4^{-1} U dt^2 + H_4 (dx + \coth \alpha_4 (H_4^{-1} - 1) dt)^2. \end{aligned} \quad (\text{A.18})$$

The total metric can therefore be written as

$$\begin{aligned} ds_{11}^2 &= H_1^{1/3} \left\{ H_1^{-1} \left[-H_4^{-1} U dt^2 + H_4 (dx + A_t dt)^2 \right] + \sum_{i=1}^4 (du^i)^2 + ds_4^2 \right\} + H_1^{-2/3} dy^2 \\ &= H_1^{-2/3} H_4 (dx + A_t dt)^2 + H_1^{-2/3} (-H_4^{-1} U dt^2 + dy^2) + H_1^{1/3} \left(\sum_{i=1}^4 (du^i)^2 + ds_4^2 \right) \end{aligned} \quad (\text{A.19})$$

with $A_t \equiv \coth \alpha_4 (H_4^{-1} - 1)$.

We can now reduce on x by reading the S-duality transformation (A.15) backwards. From the factor multiplying the first term we see that

$$e^{2\phi} = H_1^{-1} H_4^{3/2} \quad (\text{A.20})$$

and therefore

$$ds_{10}^2 = H_1^{-1} H_4^{-1/2} \left[-U dt^2 + H_4 dy^2 + H_1 H_4 \left(\sum_{i=1}^4 (du^i)^2 + ds_4^2 \right) \right]. \quad (\text{A.21})$$

The three-form gives back our B field and we also have a new one-form

$$B_{ty} = \coth \alpha_1 (H_1^{-1} - 1), \quad (\text{A.22})$$

$$A_t = \coth \alpha_4 (H_4^{-1} - 1). \quad (\text{A.23})$$

This is a two-charge solution and we only need one more.

Before boosting again, let us transform the D0-brane into a D4-brane by T-dualizing in u_1, u_2, u_3, u_4 . The dilaton becomes

$$e^\phi = \frac{H_1^{-1} H_4^{3/2}}{(H_4^{1/2})^4} = H_1^{-1} H_4^{-1/2}, \tag{A.24}$$

the B-field is unchanged and the gauge field is simply

$$A_{(5)} = \coth \alpha_4 (H_4^{-1} - 1) dt \wedge du^1 \wedge du^2 \wedge du^3 \wedge du^4. \tag{A.25}$$

The only part of the metric that changes is in the u -directions and we find

$$ds_{10}^2 = H_1^{-1} H_4^{-1/2} \left(-U dt^2 + H_4 dy^2 + H_1 \sum_{i=1}^4 (du^i)^2 + H_1 H_4 ds_4^2 \right). \tag{A.26}$$

To add the third charge we lift once more to M-theory, boost and reduce. The procedure is analogous to what has been done before and the end result is as given in section (2.2).

These U-duality transformations of the solution take place in the string frame but in the main text of the paper we always use the Einstein frame. Going to ten-dimensional Einstein frame the metric transforms as

$$g_{\mu\nu}^E = e^{-\phi/2} g_{\mu\nu}^{\text{string}} \tag{A.27}$$

and we get

$$ds_E^2 = H_1^{-\frac{3}{4}} H_4^{-\frac{3}{8}} H_0^{-\frac{7}{8}} \left(-U dt^2 + H_4 H_0 dx^2 + H_1 H_0 \sum_{i=1}^4 (du^i)^2 + H_1 H_4 H_0 \frac{L^2}{(2\pi)^2} V_{ab} dx^a dx^b \right). \tag{A.28}$$

B. Relating the c 's and \bar{c} 's

To find how the expansion coefficients of the non-extremal metric (the \bar{c} 's) are related to the original seeding solution we recall that the seeding solution has

$$-g_{tt}^{\text{seed}} \simeq U = 1 - \frac{c_t}{r}, \quad g_{zz}^{\text{seed}} \simeq 1 + \frac{c_z}{r}. \tag{B.1}$$

The harmonic functions can then be written as

$$H_a \simeq 1 + \frac{c_t}{r} \sinh^2 \alpha_a \tag{B.2}$$

and we can read the asymptotics off the new metric

$$-g_{tt} = H_1^{-\frac{3}{4}} H_4^{-\frac{3}{8}} H_0^{-\frac{7}{8}} U \tag{B.3}$$

$$= 1 - \frac{c_t}{r} \left(1 + \frac{3}{4} \sinh^2 \alpha_1 + \frac{3}{8} \sinh^2 \alpha_4 + \frac{7}{8} \sinh^2 \alpha_0 \right) + \dots \tag{B.4}$$

This shows that

$$\bar{c}_t = c_t \left(1 + \frac{3}{4} \sinh^2 \alpha_1 + \frac{3}{8} \sinh^2 \alpha_4 + \frac{7}{8} \sinh^2 \alpha_0 \right). \quad (\text{B.5})$$

Similarly we find

$$\bar{c}_x = c_t \left(-\frac{3}{4} \sinh^2 \alpha_1 + \frac{5}{8} \sinh^2 \alpha_4 + \frac{1}{8} \sinh^2 \alpha_0 \right), \quad (\text{B.6})$$

$$\bar{c}_u = c_t \left(\frac{1}{4} \sinh^2 \alpha_1 - \frac{3}{8} \sinh^2 \alpha_4 + \frac{1}{8} \sinh^2 \alpha_0 \right), \quad (\text{B.7})$$

$$\bar{c}_z = c_z + c_t \left(\frac{1}{4} \sinh^2 \alpha_1 + \frac{5}{8} \sinh^2 \alpha_4 + \frac{1}{8} \sinh^2 \alpha_0 \right), \quad (\text{B.8})$$

and

$$\bar{c}_{A_a} = -c_t \sinh \alpha_a \cosh \alpha_a. \quad (\text{B.9})$$

For the phantom direction u^0 the factor in front of $(du^0)^2$ inside the parenthesis in equation (A.28) would be $H_1 H_4$ (from the harmonic rule of [62]) and we find

$$\bar{c}_0 = c_t \left(\frac{1}{4} \sinh^2 \alpha_1 + \frac{5}{8} \sinh^2 \alpha_4 - \frac{7}{8} \sinh^2 \alpha_0 \right). \quad (\text{B.10})$$

C. Electric masses and tensions

In this appendix we examine the electric mass and tensions in greater detail.

C.1 Direct calculation

In this subsection we will show how to calculate the electric mass and tensions in 10 and 11 dimensions for general objects composed of transverse branes, F1-strings etc. obeying the harmonic function rule, and based on a seeding solution of dimension $d + 1$. This will be done using the method of equivalent sources.

The matter part of the energy momentum tensor, T^{mat} , consists of a dilatonic part, $T_{\mu\nu}^{\text{dil}}$, and parts from the gauge field strengths F_a , $T_{\mu\nu}^{(F_a)}$:

$$T_{\mu\nu}^{\text{mat}} = T_{\mu\nu}^{\text{dil}} + \sum_{a=1}^{N_{\text{ch}}} T_{\mu\nu}^{(F_a)}, \quad (\text{C.1})$$

where N_{ch} is the number of different charges. The explicit expressions for the energy-momentum tensors are:

$$8\pi G_D T_{\mu\nu}^{\text{dil}} = -\frac{1}{4} g_{\mu\nu} \partial^\rho \phi \partial_\rho \phi + \frac{1}{2} \partial_\mu \phi \partial_\nu \phi \quad (\text{C.2})$$

$$8\pi G_D T_{\mu\nu}^{(F_a)} = -\frac{1}{2} g_{\mu\nu} \frac{1}{2(p_a+2)!} e^{\kappa_a \phi} (F_a)^2 + \frac{1}{2(p_a+1)!} e^{\kappa_a \phi} (F_a)_\mu{}^{\rho_1 \dots \rho_{p_a+1}} (F_a)_{\nu \rho_1 \dots \rho_{p_a+1}}, \quad (\text{C.3})$$

where D is the total number of dimensions, p_a is the number of world-volume directions for object a , and κ_a is a number depending on p_a .

To calculate the mass and tension we use the method of equivalent sources. This means that instead of studying the real metric we will study a metric with the same asymptotics as our solution, but which is everywhere Newtonian, i.e. we can split the metric as:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{C.4}$$

where $\eta_{\mu\nu}$ is the Minkowski metric and $h_{\mu\nu}$ is small so we can ignore second order contributions in h .

We now define:

$$S_{\mu\nu} \equiv T_{\mu\nu} - \frac{1}{D-2}g_{\mu\nu}T \simeq T_{\mu\nu} - \frac{1}{D-2}\eta_{\mu\nu}T, \tag{C.5}$$

where D is the total number of dimensions and $T = T^\mu{}_\mu \simeq \eta^{\mu\nu}T_{\nu\mu}$. We can inverse this as:

$$T_{\mu\nu} \simeq S_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}S. \tag{C.6}$$

Then Einstein's equation takes the form

$$S_{\mu\nu}^{\text{mat}} = \frac{1}{8\pi G_D}R_{\mu\nu} \tag{C.7}$$

The linearized Ricci tensor is

$$R_{\mu\nu}^{(1)} = -\frac{1}{2}\left(\square h_{\mu\nu} + h_{\lambda}{}^\lambda{}_{,\mu\nu} - h_{\mu}{}^\lambda{}_{,\nu\lambda} - h_{\nu}{}^\lambda{}_{,\mu\lambda}\right) \tag{C.8}$$

The gravitational part of the energy-momentum tensor is then defined from:

$$S_{\mu\nu}^{\text{gr}} \equiv \frac{1}{8\pi G_D}\left(R_{\mu\nu}^{(1)} - R_{\mu\nu}\right) \tag{C.9}$$

Summing the contributions from both the gravitational part and the matter part gives:

$$S_{\mu\nu} = S_{\mu\nu}^{\text{gr}} + S_{\mu\nu}^{\text{dil}} + \sum_{a=1}^{N_{\text{ch}}} S_{\mu\nu}^{(F_a)} = \frac{1}{8\pi G_D}R_{\mu\nu}^{(1)} \tag{C.10}$$

The electric part of the energy momentum tensor will simply be defined as the part of the tensor that goes to zero when we set the charges to zero, i.e.:

$$S_{\mu\nu}^{\text{el}} \equiv S_{\mu\nu} - S_{\mu\nu}|_{Q_a=0} \tag{C.11}$$

We note that we a priori have contributions from all the parts of the energy momentum tensor in (C.10), especially, we see that both the terms from the gauge fields and the dilatonic term (the dilaton is constant in the case where the charges are zero) are completely electric.

Using that all raisings and lowerings can be done with $\eta_{\mu\nu}$, that the covariant derivatives can be replaced with ordinary, and that $h_{\mu\nu}$ only depends on r and is diagonal, we get from (C.6):

$$\begin{aligned} T_{\mu\mu} &= \frac{1}{16\pi G_D}\partial_r^2\left(-h_{\mu\mu} - \eta_{\mu\mu}(h_{rr} - \eta^{\alpha\beta}h_{\beta\alpha})\right) \\ &= \partial_r^2\left(-h_{\mu\mu} - \eta_{\mu\mu}\left(h_{tt} - \frac{(d-2)}{r^2}h_{\Omega\Omega} - h_{zz} - \sum_a p_a h_{u_{(a)}u_{(a)}}\right)\right), \end{aligned} \tag{C.12}$$

where $\mu \neq r$, Ω refers to the angular directions, and $u_{(a)}^i$ are the coordinates for the world-volume for object a which will have a volume denoted by V_{p_a} . We see that this is a boundary term so that we easily can get the masses and tensions (since $h_{xx} \sim \bar{c}_x/r^{d-3}$):

$$\bar{M} = \int T_{tt} = \frac{(\prod V_{p_a})L\Omega_{(d-2)}}{16\pi G_D}(d-3)\left((d-2)\bar{c}_\Omega + \bar{c}_z + \sum_a p_a \bar{c}_{u_{(a)}}\right), \quad (\text{C.13})$$

$$L\bar{T}_z = - \int T_{zz} = \frac{(\prod V_{p_a})L\Omega_{(d-2)}}{16\pi G_D}(d-3)\left((d-2)\bar{c}_\Omega + \sum_a p_a \bar{c}_{u_{(a)}} - \bar{c}_t\right), \quad (\text{C.14})$$

$$L_{u_{(a)}}\bar{T}_a = \frac{(\prod V_{p_a})L\Omega_{(d-2)}}{16\pi G_D}(d-3)\left((d-2)\bar{c}_\Omega + \bar{c}_z + \sum_{a' \neq a} p_{a'} \bar{c}_{u_{(a')}} + (p_a - 1)\bar{c}_{u_{(a)}} - \bar{c}_t\right). \quad (\text{C.15})$$

The asymptotic quantities are determined by the real physical metric which by the harmonic function rule has components:

$$g_{tt} = -H\left(\prod_a H_{(a)}^{-1}\right)U, \quad g_{u_{(a)}^i u_{(a)}^i} = H_{(a)}^{-1}H, \\ g_{zz} = Hf_z, \quad g_{\Omega\Omega} = Hf_\Omega, \quad (\text{C.16})$$

where $U \sim 1$, $f_z \sim 1$, and $f_\Omega \sim r^2$ are functions from the seeding solution, and:

$$H \equiv \prod_a H_{(a)}^{\beta_a}, \quad \beta_a = \frac{p_a + 1}{D - 2}, \quad (\text{C.17})$$

which holds for Dp-branes, F1-strings and NS5-branes in $D = 10$ dimensions, and M2- and M5-branes in $D = 11$ dimensions. The harmonic function is assumed to be given by:

$$H_{(a)} = 1 + (1 - U) \sinh^2 \alpha_a. \quad (\text{C.18})$$

The electric part of the mass and tensions are defined by (C.11) as the part that goes to zero when we set the charges to zero. We can now use the metric to find the electric parts of \bar{c}_t , \bar{c}_z , etc. in terms of the seeding c_t :

$$\bar{c}_t^{\text{el}} = \sum_a \frac{D - 2 - p_a - 1}{D - 2} \sinh^2 \alpha_a c_t \\ \bar{c}_{u_{(a)}}^{\text{el}} = \sum_{a'} \frac{p_{a'} + 1}{D - 2} \sinh^2 \alpha_{a'} c_t - \sinh^2 \alpha_a c_t \\ \bar{c}_z^{\text{el}} = \sum_a \frac{p_a + 1}{D - 2} \sinh^2 \alpha_a c_t \\ \bar{c}_\Omega^{\text{el}} = \bar{c}_z^{\text{el}} \quad (\text{C.19})$$

Inserting this in (C.13)–(C.15) finally gives:

$$\bar{M}^{\text{el}} = \frac{(\prod V_{p_a})L\Omega_{(d-2)}}{16\pi G_D}(d-3)c_t \sum_a \sinh^2 \alpha_a \quad (\text{C.20})$$

$$L\bar{T}_z^{\text{el}} = 0 \quad (\text{C.21})$$

$$L_{u_{(a)}}\bar{T}_a^{\text{el}} = \frac{(\prod V_{p_a})L\Omega_{(d-2)}}{16\pi G_D}(d-3)c_t \sinh^2 \alpha_a \quad (\text{C.22})$$

which, of course, is the same result as (2.30). We note that since we should look at the asymptotic behavior of our metric, then by the form of $H_{(a)}$ in (C.18) our electric masses and tensions split in a sum of contributions from each object a . We also observe that there is no contribution to the electric parts of the tension in some given direction from objects transverse to this direction. Especially, the electric part of the tension in the z -direction is zero. We will use this as one of the basic principle in the next section.

C.2 Symmetry considerations

In this section we will investigate the electric masses and tensions, but this time based on some simple symmetry consideration and some physical principles that have been confirmed in the last subsection. We will follow the analysis in [11, section 3.2], but with our definition of the electric mass and tension.

First, we assume that the electric parts of the energy-momentum tensor split up into contributions from each of the objects a , as was confirmed in last section, i.e.:

$$T_{\mu\nu}^{\text{el}(a)} = T_{\mu\nu}|_{\forall a' \neq a: Q^{a'}=0} - T_{\mu\nu}|_{\forall a': Q^{a'}=0}. \tag{C.23}$$

This gives rise to the electric parts of the mass and tensions.

We will still use the method of equivalent sources, so that we can neglect second order contributions in $h_{\mu\nu}$. We then require boost-invariance for object a , i.e. $T_{tt}^{\text{el}(a)} = -T_{u_{(a)}^j u_{(a)}^j}^{\text{el}(a)}$. This can be seen to be fulfilled by the dilatonic and gauge part of the energy-momentum tensor in (C.2) and (C.3) (using that ϕ only depends on r) and, actually, also for $R_{\mu\nu}^{(1)}$ using (C.16) (assuming the symmetry holds for the equivalent metric also) and (C.8), and hence for the whole energy-momentum tensor. After integrating, the boost-invariance implies:

$$M^{\text{el}(a)} = L_{u_{(a)}} \bar{T}_a^{\text{el}(a)}. \tag{C.24}$$

We further require that for ν a transverse direction to object a we have (i.e. after integrating the electromagnetic part of the energy-momentum tensor):

$$L_\nu \bar{T}_\nu^{\text{el}(a)} = - \int T_{\nu\nu}^{\text{el}(a)} = 0, \quad \nu \text{ transverse to object } a. \tag{C.25}$$

This means that we in (C.24) can replace the tension on the right hand side with the total electric tension in the $u_{(a)}$ -direction:

$$M^{\text{el}(a)} = L_{u_{(a)}} \bar{T}_a^{\text{el}}. \tag{C.26}$$

Assuming a diagonal energy-momentum tensor and using the method of equivalent sources we get from [44] that

$$\nabla^2 g_{u_{(a)}^i u_{(a)}^i} = -16\pi G_D \left(T_{u_{(a)}^i u_{(a)}^i} - \frac{1}{D-2} T^\rho_\rho \right). \tag{C.27}$$

Taking the non-electric part of this we should set all the charges to zero. In that case the directions $u_{(a)}^i$ should be completely flat, and hence the left hand side should be zero, i.e.:

$$T_{u_{(a)}^i u_{(a)}^i}^{\text{non-el}} = \frac{1}{D-2} \sum_\rho \eta^{\rho\rho} T_{\rho\rho}^{\text{non-el}}, \tag{C.28}$$

or:

$$(D - 2)T_{u_{(a)}u_{(a)}}^{\text{non-el}} = -T_{tt}^{\text{non-el}} + T_{zz}^{\text{non-el}} + \sum_a p_a T_{u_{(a)}u_{(a)}}^{\text{non-el}} + \text{overall transverse terms.} \quad (\text{C.29})$$

Integrating, using (C.25) and (C.26), and solving we get:

$$(d - 1)(L_{u_{(a)}}\mathcal{T}_a - M^{\text{el}(a)}) = \bar{M} - M^{\text{el}} + L\bar{\mathcal{T}}_z. \quad (\text{C.30})$$

In general these N_{ch} equations can be solved for the unknowns $M^{\text{el}(a)}$. However, precisely in our three-charge case with $d - 1 = 3$ the equations have determinant zero. Adding the three equations gives:

$$L_x\bar{\mathcal{T}}_x + L_{u^i}\bar{\mathcal{T}}_{u^i} + L_0\bar{\mathcal{T}}_0 = \bar{M} + L\bar{\mathcal{T}}, \quad (\text{C.31})$$

which exactly is independent of M^{el} . From this equation we, however, obtain the two nice relations:

$$\bar{n}_x + \bar{n}_0 + \bar{n}_u = 1 + \bar{n}, \quad (\text{C.32})$$

$$r_x + r_0 + r_u = 1 + r \quad (\text{C.33})$$

which are indeed obeyed by (2.34) and (2.35), and (4.30).

For general dimensions and charges we instead get:

$$\sum_a \bar{n}_a = \frac{N_{\text{ch}}}{d-1} + \frac{N_{\text{ch}}}{d-1}\bar{n}, \quad (\text{C.34})$$

$$\sum_a r_a = \left(1 - \frac{d-2}{2}\right) \frac{N_{\text{ch}}\mu}{(d-1)\epsilon} + \frac{3N_{\text{ch}}}{2(d-1)}r, \quad (\text{C.35})$$

where in the last equation the limit $\alpha_a \rightarrow \infty$ had to be taken in the same way as in the end of section 4.3. We see that for $d = 4$ the last relation reduces to:

$$\sum_a r_a = \frac{N_{\text{ch}}}{2}r. \quad (\text{C.36})$$

If we use this for the three-charge case we exactly get $r = 2$.

Finally, if we use this on near-extremal two-charge case we get:

$$r_0 + r_4 = r \quad (\text{C.37})$$

in agreement with (6.3) (we set the last charge to zero). Using also (C.33) we also conclude that the last relative tension should be one again in agreement with (6.3).

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